Estimation of Multicomponent Stress-Strength Reliability Based on Lower Record Values from the Topp-Leone Distribution

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Introduction and Motivation

In reliability theory, the stress–strength model is defined by $R = P(X \geq Y)$, where X and Y are strength and stress, respectively.

Multicomponent stress–strength (MSS) models: A system with k identical and independent components that works if at least s $(1 \leq s \leq k)$ of the components survive a common stress.

Lower Record Values: Only extrem minimum observations are recorded.

TL Distribution and Records

Topp–Leone distribution

$$f(x,\alpha) = 2\alpha(1-x)\left[x(2-x)\right]^{\alpha-1}, \qquad 0<x<1, \quad \alpha>0, \tag{1}$$

$$F(x,\alpha) = \left[x(2-x)\right]^{\alpha}, \qquad 0<x<1, \quad \alpha>0, \tag{2}$$

Lower Record Values:

An observation X_j is called a lower record if $X_j < X_i$ for all $i<j$. The probability joint density function of $\underline{U} = (U_1, U_2, ..., U_n)$ is given by

$$f_{\underline{U}}(u_1, u_2, \ldots, u_n) = \prod_{i=1}^{n-1} \frac{f(u_i)}{F(u_i)} f(u_n).$$

MSS Reliability Formula

$$R_{s,k} = \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{+\infty} (1 - F(y))^i F^{k-i}(y)\, dG(y). \tag{3}$$

For TL Distribution with parameters α_1 (strenght) and α_2 (stress):

$$R_{s,k} = \rho \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} (k + \rho - j) \right]^{-1}, \quad \rho = \frac{\alpha_2}{\alpha_1} \tag{4}$$

Maximum Likelihood Estimation (MLE)

Let $\underline{u} = (u_1, u_2, \ldots, u_n)$ be the first n lower record values from $TL(\alpha_1)$, and let $\underline{v} = (v_1, v_2, \ldots, v_m)$ be the first m lower record values from $TL(\alpha_2)$. Explicit MLES:

$$\hat{\alpha}_1 = \frac{n+1}{-\ln[u_n(2-u_n)]}, \tag{5}$$

$$\hat{\alpha}_2 = \frac{m+1}{-\ln[v_m(2-v_m)]}. \tag{6}$$

Hence, the MLE of $R_{s,k}$:

$$\hat{R}_{s,k} = \hat{\rho} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} (k + \hat{\rho} - j) \right]^{-1}, \;\; \hat{\rho} = \frac{\hat{\alpha}_2}{\hat{\alpha}_1} \tag{7}$$

Asymptotic Variance

$$V(\hat{R}_{s,k}) = \sigma_{11}\left(\frac{\partial \hat{R}_{s,k}}{\partial \hat{\alpha}_1}\right)^2 + \sigma_{22}\left(\frac{\partial \hat{R}_{s,k}}{\partial \hat{\alpha}_2}\right)^2 \tag{8}$$

Asymptotic confidence interval 95% : $\hat{R}_{s,k} \pm Z_{1-\frac{\alpha}{2}}\sqrt{V(\hat{R}_{s,k})}$.

Bootstrap Methods: Percentile (Boot-p) and (Boot-t)

Bayesian Estimation via MCMC

Let $\alpha_1 \sim \Gamma(a_1, b_1)$ and $\alpha_2 \sim \Gamma(a_2, b_2)$.

Bayesian Estimator (SEL):

$$\tilde{R}_{s,k}^{Bayes} = \int_0^\infty \int_0^\infty R_{s,k}\pi^*(\alpha_1, \alpha_2 | \underline{u}, \underline{v}) d\alpha_1 d\alpha_2. \tag{9}$$

computation: Metropolis-Hastings (MCMC) with normal proposal.

Table: The Bias and MSE for estimate of $R_{s,k}$, results for $(\alpha_1, \alpha_2) = (0.5, 2)$ and $(s, k) = (1, 3)$ and Real $R_{s,k} = 0.4286$

(n, m)	MLE		Prior I		Prior II	
	AB	MSE	AB	MSE	AB	MSE
(5, 5)	0.123	0.0228	0.1074	0.0181	0.1011	0.0162
(10, 10)	0.0843	0.0111	0.0804	0.0101	0.0761	0.0092
(15, 15)	0.0632	0.0065	0.0671	0.0074	0.064	0.0068

conclusion: Bayesian estimates (especially Prior II) perform better than MLE by comparing ABs and MSEs and the same result for real data.

Conclusions

ML and Bayes estimators of $R_{s,k}$ were derived using lower record values. Asymptotic, boot-P, boot-t and HPD credible intervals were constructed. Bayesian estimation with informative priors (Prior II) perform better than MLE Topp leaon distribution is suitable for lifetime data.

Future work: Extend to upper record.

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