دانشگاه فردوسی مشهد

دانشکده علوم ریاضی و آمار

Maximum Varma entropy distribution
With
conditional known Lorenz curve constraints

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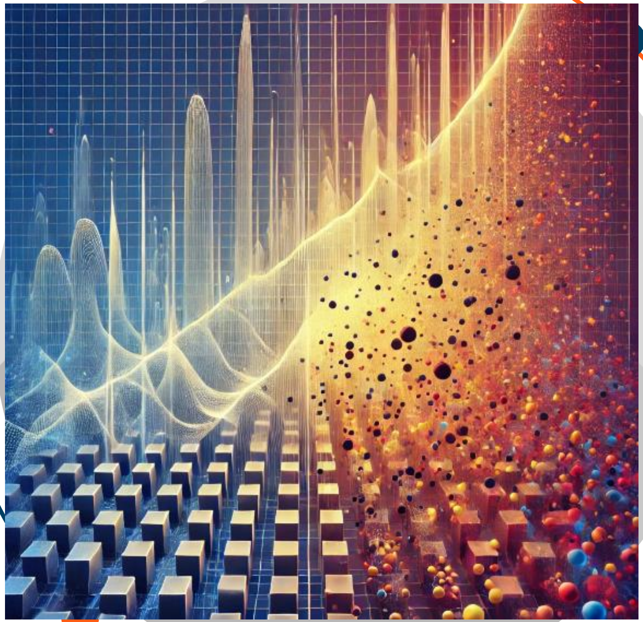
Introduction



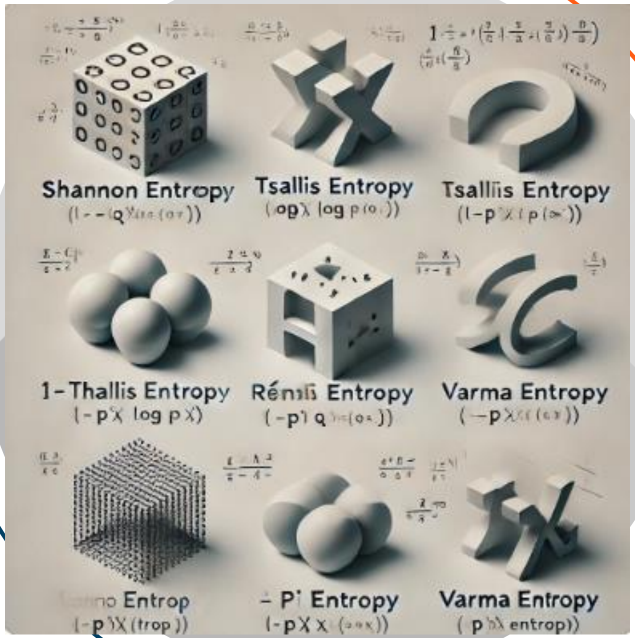
Among all distributions consistent with known constraints, choose the one with maximum entropy—this introduces the least bias and avoids unfounded assumptions.

Introduced by Jaynes, this principle is widely applied in statistics, physics, information theory, and economics.

A common use is modeling income and wealth distribution across populations.

Maximum Varma entropy distribution With conditional known Lorenz curve constraints

Introduction



To gain flexibility in modeling complex systems, we use Varma entropy—a generalized entropy measure that includes many classic entropies as special cases.

We incorporate the Lorenz curve as a key constraint, alongside the mean, to directly capture income inequality in our maximum entropy framework.

The resulting distribution is empirically tested on real income data from two countries and compared with exponential and Fréchet benchmarks using MSE and MAE.

This approach offers a broader, data-adaptive framework for modeling economic distributions.

Preliminaries

- Shannon entropy is the standard measure, defined for a continuous random variable X with probability density function f

$$H(f) = -\int_0^\infty f(x) log f(x)$$

- Introduced by Varma in 1966 as a generalization of standard entropy measures in statistical mechanics. Defined for a continuous random variable X with probability density function f, it extends Shannon's framework for broader applicability.

$$H_{\alpha,\beta}(f(x)) = \frac{1}{\beta-\alpha} ln\left[\int_0^\infty (f(x))^{\alpha+\beta-1} dx\right]$$

The Lagrangian Functional Formulation

A key technique in calculus of variations is optimizing functionals of the form

$$L(y) = \int_a^b G(y, y', x)dx$$

where G is smooth.

The Euler–Lagrange equation provides a necessary condition for an extremum:

$$\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'} = 0$$

Inequality measures (Lorenz Curve)

Inequality is commonly studied using indicators derived from income or wealth distributions.

The Lorenz curve, introduced by Lorenz, is a key graphical tool defined for a non-negative random variable X with CDF F and mean μ.

$$L_F(p) = \frac{1}{\mu}\int_0^p F^{-1}(x)dx$$

Motivation for Lorenz-Constrained Entropy

Moment constraints summarize basic statistical properties but fail to capture the full structure of economic inequality.

The Lorenz curve offers detailed cumulative distribution information, making it a more informative constraint.

We derive the maximum Varma entropy distribution under joint constraints of the Lorenz curve and the mean.

This yields a theoretically grounded distribution that reflects observed inequality while remaining consistent with the empirical mean.

کاربرد در قابلیت اعتماد

- normalization condition

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

- global mean constraint

$$\int_{-\infty}^{\infty} xf(x)dx = \mu$$

- tail probability

$$\int_{-\infty}^{F^{-1}(p)} f(x)dx = \xi$$

- expected shortfall

$$\int_{-\infty}^{F^{-1}(p)} xf(x)dx = \nu$$

$$g^T(x) = \begin{cases} 1 & x \leq F^{-1}(p) \\ 0 & x \geq F^{-1}(p) \end{cases} \qquad g^C(x) = \begin{cases} x & x \leq F^{-1}(p) \\ 0 & x \geq F^{-1}(p) \end{cases}$$

Applying the Lagrange equation yields the general maximum Varma entropy distribution.

$$L = \int_{-\infty}^{\infty} G(F(x), F'(x), x)dx$$

$$= \frac{1}{\beta - \alpha} ln \left[\int_{-\infty}^{\infty} (f(x))^{\alpha+\beta+1}\, dx\right]$$

$$- \lambda_0 \int_{-\infty}^{\infty} f(x)dx - \lambda_1 \int_{-\infty}^{\infty} g^T(x) f(x)dx - \lambda_2 \int_{-\infty}^{\infty} xf(x)dx - \lambda_3 \int_{-\infty}^{\infty} g^C(x) f(x)dx$$

With attend to situation in above, we obtain the probability density distribution under the constraints

$$f(x) = \left(\overline{(\lambda_0\ )} + \overline{(\lambda_1\ )}\, g^T\ (x) + \overline{(\lambda_2\ )}x + \overline{(\lambda_3\ )}g^C\ (x)\right)^{\frac{1}{\alpha+\ \beta-2}}$$

$\overline{\lambda_0}\ \ldots\overline{\lambda_3}$.: normalized Lagrange multipliers derived from the mean and Lorenz constraints.

These yield the explicit maximum Varma entropy density.

The proposed maximum Varma entropy distribution is applied to 1988 decile income data for Malaysia and Singapore (from Chotikapanich).

The Lorenz curve is used, focusing on the bottom 10% income share ($p=0.1$) as a key constraint.

The distribution is fitted to both countries' data under the constraints of the known mean and Lorenz curve.

The maximum Varma entropy distribution consistently outperforms both exponential and Fréchet benchmarks, achieving lower MSE and MAE values for Malaysia and Singapore.
The improvement is especially notable for Malaysia, where the Fréchet distribution shows very large errors, while the proposed model remains accurate.
These findings confirm that incorporating the Lorenz curve along with the mean constraint within the Varma entropy framework yields a distribution that better fits observed income data.
The proposed approach offers a superior alternative for modeling income distributions with significant inequality.

	Maximum entropy distribution		Exponential distribution		Fréchet distribution	
Country	MSE	MAE	MSE	MAE	MSE	MAE
Singapore	0.02	0.12	0.28	0.39	2.93	0.79
Malazaya	0.05	0.23	0.29	0.49	>10	>10

Conclusion

This study developed a maximum entropy framework using Varma entropy under the constraints of the mean and the Lorenz curve, directly embedding both average level and inequality structure into the distribution.

Empirical testing on 1988 decile income data for Malaysia and Singapore shows that the proposed distribution consistently outperforms exponential and Fréchet benchmarks in terms of MSE and MAE.

The improvement is especially notable for Malaysia, where the Fréchet model performs poorly, while the proposed approach remains accurate.

These results confirm that incorporating Lorenz curve information enhances the model's ability to fit real income data.

Overall, the maximum Varma entropy approach offers an effective and robust alternative for modeling income distributions with inequality.

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