A Statistical Model for Real Lifespan Under Environmental Conditions: A Gamma-Dirichlet Approach

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Introduction and Motivation

- Traditional laboratory-based lifespan assessments often fail to capture real-world environmental complexities (temperature fluctuations, mechanical stress, etc.).
- **Objective:** Develop a statistical framework incorporating environmental variability.
- **Our Approach:** Define "real lifetime" as the product of:
  1. A laboratory-measured baseline lifetime.
  2. An environmental coefficient vector modulating external conditions.
- **Applications:** Reliability analysis in civil engineering, manufacturing, and project management.

Preliminaries and Definitions

Definition: Real Lifetime

Let $Y \sim \text{Gamma}(\alpha, \beta)$ be the *laboratory lifetime* and $\mathbf{X} \sim \text{Dir}(\alpha_1, \dots, \alpha_k)$ be the *environmental coefficient vector* $(\sum X_i = 1)$.
The *real lifetime* $\mathbf{Z}$ is the component-wise product:

$$\mathbf{Z} = \mathbf{X}Y = (X_1 Y, X_2 Y, \dots, X_k Y)$$

Definition: Aggregate Real Lifetime

For r independent positions, the aggregate real lifetime is:

$$\mathbf{T} = \sum_{i=1}^{r} \mathbf{X}_i Y_i$$

Distribution of the Real Lifetime

Theorem: Joint PDF

Let $\mathbf{X} \sim \mathrm{Dir}(\alpha_1, \ldots, \alpha_r)$ and $Y \sim \mathrm{Gamma}(\alpha, \beta)$ be independent. The joint PDF of $\mathbf{Z} = (Z_1, \ldots, Z_r)$ is:

$$f(z_1, \ldots, z_r) = \frac{\beta^{-\alpha}\Gamma\left(\sum \alpha_i\right)}{\Gamma(\alpha)\prod \Gamma(\alpha_i)} \left(\sum z_i\right)^{\alpha - \sum \alpha_i} e^{-\frac{\sum z_i}{\beta}} \prod z_i^{\alpha_i - 1}$$

Corollary: Independence

If $\alpha = \sum_{i=1}^{r} \alpha_i$, the components $Z_1, \ldots, Z_r$ are **mutually independent**, and each follows $\mathrm{Gamma}(\alpha_i, \beta)$.

Moments and Covariance Structure

Theorem: Marginal Expectations and Covariance

For the real lifetime vector $\mathbf{Z}$, the moments are:

$$\mathbb{E}(Z_i) = \frac{\alpha_i}{\sum \alpha_j}\alpha\beta$$

$$\mathbb{V}\mathrm{ar}(Z_i) = \frac{\alpha\beta^2\alpha_i}{(\sum \alpha_j)^2}\left[(\alpha+1)(\alpha_i+1) - \frac{\alpha\alpha_i(\sum \alpha_j + 1)}{\sum \alpha_j}\right]$$

$$\mathbb{C}\mathrm{ov}(Z_i, Z_j) = \frac{\alpha\alpha_i\alpha_j\beta^2}{(\sum \alpha_k)^2(\sum \alpha_k + 1)}\left(\alpha + 1 - \frac{\alpha(\sum \alpha_k + 1)}{\sum \alpha_k}\right)$$

Extension to Multiple Independent Positions

Theorem: Aggregate Mean and Variance

For the aggregate lifetime $\mathbf{T} = \sum_{i=1}^{r} \mathbf{X}_i Y_i$ with independent components:

$$\mathbb{E}(\mathbf{T}) = \sum_{i=1}^{r} \alpha_i \beta_i \boldsymbol{\mu}_i$$

$$\mathbb{V}\mathrm{ar}(\mathbf{T}) = \sum_{i=1}^{r} \left[\alpha_i \beta_i^2 \boldsymbol{\Sigma}_i + \alpha_i \beta_i^2 \boldsymbol{\mu}_i \boldsymbol{\mu}_i'\right]$$

where $\boldsymbol{\mu}_i = \mathbb{E}(\mathbf{X}_i)$ and $\boldsymbol{\Sigma}_i = \mathbb{V}\mathrm{ar}(\mathbf{X}_i)$.

Approximation Method

- Exact distributions can be mathematically intractable for complex systems.
- **Proposed Method:** Moment matching to a Beta distribution.
- **Simulation Results:** For $Z = UV/W$ $(U, V, W \sim \mathcal{U}[0, 1])$, matching moments yields $\text{Beta}(1.78, 1.71)$.
- **Goodness-of-Fit:** Kolmogorov-Smirnov tests show consistently high p-values (0.50 to 0.97) across various Dirichlet and Gamma parameter configurations, validating the Beta approximation.

Real Data Analysis: Tabriz Tractor Mfg. Co.

- **Dataset:** $n = 150$ lifetime vectors under 3 environmental conditions.
- **Empirical Means:** Condition 1: 1542h, Condition 2: 2109h, Condition 3: 1130h.
- **Model Fitting:** Using method-of-moments, we estimated:

$$\hat{\alpha} = 5.19, \quad \hat{\beta} = 921, \quad \hat{\boldsymbol{\alpha}}_X = (1.91, 2.59, 1.39)$$

- **Validation:** The fitted model reproduces observed means with $< 0.2\%$ error. KS test on normalized vectors confirms Dirichlet assumption (p-value $= 0.73$).

Conclusion and Future Work

Conclusion

- Introduced a Gamma-Dirichlet framework for modeling product lifespan under environmental conditions.
- Derived joint distributions, moments, and aggregate properties.
- Validated a practical Beta approximation and applied the model to real industrial data.

Future Work

- Direct multivariate approximation of the Dirichlet mixture.
- Extensions to non-parametric formulations and covariate-dependent environmental factors.

References

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