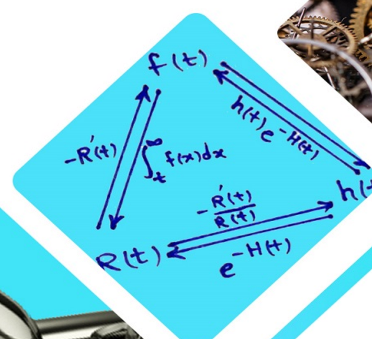
Seminar Schedule and Abstracts Book

9th Seminar on Reliablity Theory and its applications

Isfahan University of Technology
May 24-25 2023



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In the Name of Allah

Abstracts of

The 9th Seminar on

Reliability Theory and its Applications

Department of Mathematical Sciences, Isfahan University of Technologgy,
Isfahan, Iran

and

Ordered Data, Reliability and Dependency Center of Excellence

Ferdowsi University of Mashhad,

Mashhad, Iran

May 24-25, 2023

Disclaimer

This book contains the abstracts booklet of the 9th Seminar on "Reliability Theory and its Applications". Authors are responsible for the contents and accuracy. Opinions expressed may not necessarily reflect the position of the scientific and organizing committees.

Preface

Continuing the series of workshops on "Reliability Theory" in the Ferdowsi University of Mashhad and Eight Seminars at the University of Isfahan (2015), University of Tehran (2016), Ferdowsi University of Mashhad (2017), Shiraz University (2018), Yazd University (2019), University of Mazandaran (2020), University of Birjand (2021), and Ferdowsi University of Mashhad (2022) we are pleased to organize virtually (online) the 9th Seminar on Reliability Theory and its Applications" during 24-25 May 2023 at the Department of Mathematical Sciences, Isfahan University of Technology. On behalf of the organizing and scientific committees, we would like to extend a very warm welcome to all participants in this event. We hope that this seminar provides an environment of useful discussions and will also exchange scientific ideas through opinions. We wish to express our gratitude to the numerous individuals and organizations that have contributed to the success of this seminar, in which around 100 colleagues, researchers, and postgraduate students have participated. Finally, we would like to extend our sincere gratitude to the administration of the Isfahan University of Technology, the Department of Mathematical Sciences, Ferdowsi University of Mashhad, the "Ordered Data, Reliability, and Dependency" Center of Excellence, the Iranian Statistical Society, the Mathematic House, Islamic World Science Citation Database (ISC), Snowa Tech Company, the Scientific Committee, the Executive Committee, and the students of the Department of Statistics at the Isfahan University of Technology, for their kind cooperation.

The Organizing Committee
May, 2023

Topics

The aim of the seminar is to provide a forum for presentation and discussion of scientific works covering theories and methods in the field of reliability theory and its applications in a wide range of areas:

- Lifetime distributions theory
- Accelerated life testing
- Maintenance modeling and analysis
- Reliability of systems
- Stochastic orderings in reliability
- Networks reliability
- Survival analysis
- Bayesian methods in reliability
- Case studies in reliability analysis
- Stress-strength modeling
- Shock models in reliability
- Optimization methods in reliability
- Lifetime data analysis
- Stochastic processes in reliability
- Data mining in reliability
- Computational algorithms in reliability
- Stochastic dependency in reliability
- Safety and risk assessment
- Degradation models
- Software reliability
- Stochastic aging
- Warranty models

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Contents

Estimation of parameters of Kumaraswamy distribution from progressively Type-I interval censored data using EM algorithm

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Abstract

EM algorithm is used to derive the maximum likelihood estimates of unknown parameters when the gathered data are progressively Type-I interval censored. It is assumed that the lifetimes follow a Kumaraswamy distribution. Finally, a simulated data set is analyzed for demonstrative purposes.

Keywords: EM algorithm, Kumaraswamy distribution, Maximum likelihood estimate, Progressive Type-I interval censoring scheme.

1 Introduction

In reliability and life testing experiments, the failure times of all of units on the test due to the time limitation or other restrictions may not be observed exactly. Also, in industrial life testing and medical survival analysis, it is very often that the object is lost or withdrawn before failure. Hence, the problem of occuring censored observations is quite commonly when observing lifetime data. There are various types of censoring schemes. Among them, Type-I and Type-II censoring schemes are the most common for considerations; see, for example, David and Nagaraja [1]. Uner Type-I censoring, the test ceases at a pre-fixed time and under

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Type-II censoring, the test terminates a pre-fixed number of failures. But, these two censoring schemes do not allow for units to be removed from the test at points other than the final termination point. When such an allowance is desirable, progressive censoring is suggested. For a comprehensive review of progressive censoring, we refer the readers to Balakrishnan and Aggarwala [2].

In practice, it is often impossible to inspect the lifetime testing process continuously, even with censoring. But, the periodic inspection of the units on the test is feasible. That is, the experimenter can only count the number of failures in a time interval instead of measuring failure times exactly. Such a life test is called interval censoring. Combining progressive censoring and interval censoring, a new censoring introduced by Aggarwala [3] and called progressive Type-I interval censoring is obtained which is described briefly as follows. Suppose that n units are placed on a test at time zero. Let $t_1 < \cdots < t_m$ be the pre-scheduled inspection times, where t_m is the termination time for the test. At the i-th inspection time t_i, the number X_i of units failed in the interval $(t_{i-1}, t_i]$ is recorded and then R_i of surviving units are randomly removed from the remaining units $n - \sum_{j=1}^{i} X_j - \sum_{j=1}^{i-1} R_j$ for $i = 1, \ldots, m$. Since the number of surviving units at inspection time t_i, say Y_i, is a random variable and the number of units removed from the test should not be greater than Y_i, R_i can be a pre-determined percentage, say p_i, of the remaining surviving units at time t_i. For example, under the pre-determined percentage values $p_1, \ldots, p_k$ ($p_k = 1$), the number of units removed at time t_i can be considered as $R_i = \lfloor p_i Y_i \rfloor$, where $\lfloor w \rfloor$ is the largest integer which is smaller than or equal to w. Alternatively, assuming $R_1, \ldots, R_m$ to be pre-determined non-negative integers, the number of surviving units removed from the test can be considered as $R_i^* = \min(R_i, \text{number of remaining units at time } t_i)$ for $i = 1, \ldots, m-1$ and $R_m^* =$ number of surviving units at time t_m. When $R_1 = \cdots = R_{m-1} = 0$ and so $R_m = n - \sum_{i=1}^{m} X_i$, this censoring scheme is called the Type-I interval censoring.

The progressively Type-I interval censored data have been considered by many authors for the parametric inferences on different distributions. For example, Ashour and Afify [4], Arabi Belaghi et al. [5], Lin et al. [6], Chen et al. [7], Ng and Wang [8], Teimouri and Gupta [9], Singh and Tripathi [11], Du et al. [11] and Teimouri [12] compared different estimation methods when the available data are progressively Type-II censored data coming from the distributions such as exponentiated Weibull, Burr XII, log-normal, generalized exponential, Weibull, Gompertz-Makeham, inverse Weibull, log-logistic and Chen, respectively.

Although progressive Type-I interval censored sampling is very applicable in lifetime experiments, not much attention has been paid to due to the complicated calculations of the corresponding likelihood function. The expectation-maximization (EM) algorithm is one of the standard methods to obtain a stationary point of the likelihood function. This algorithm is a very powerful tool in determining the maximum likelihood (ML) estimates of the unknown parameters on the basis of the incomplete data such as progressively Type-I interval censored data. Unlike the more traditional methods of computing the ML estimates such as the Newton-Raphson method which are very sensitive to their initial parameter estimation values, the EM algorithm is relatively robust against these values. Also, with the EM algorithm, it is not required to calculate the first and second derivatives of the log-likelihood function. Usually the calculations of the derivatives on the basis of the progressively censored data are complicated. For more details on the EM algorithm, the readers may refer to Dempster et al. [13] and McLachlan and Krishnan [14].

In this paper, we assume that the lifetimes follow the Kumaraswamy distribution proposed by Kumaraswamy [15]. This distribution is suitable for many natural phenomena whose outcomes has lower and upper bound, for example, the heights and weights of individuals, scores obtained on a test, atmospheric temperatures. The probability density function (PDF) and cumulative distribution function (CDF) of the Kumaraswamy distribution with the first and second shape parameters α and λ, respectively, are as follows

$$f(x) = \alpha\lambda x^{\lambda-1}\left(1-x^{\lambda}\right)^{\alpha-1}, \quad 0<x<1,\ \alpha,\lambda>0, \tag{1}$$
$$F(x) = 1-\left(1-x^{\lambda}\right)^{\alpha}, \qquad 0<x<1,\ \alpha,\lambda>0, \tag{2}$$

respectively. It can be shown that the failure rate function of the Kumaraswamy distribution is an increasing function. Also, the Kumaraswamy distribution as a lambda-type distribution is supported on the interval $(0,1)$. Jones [16] discussed the similarities and differences between the lambda and Kumaraswamy distributions. Some of the advantages of the Kumaraswamy distribution are that the CDF of this distribution has a closed form, quantiles of this distribution are easily attainable and one can easily generate random variables from this distribution. The Kumaraswamy distribution attracted much attention in recent years. Seifi et al. [17] proposed a method for maximizing the manufacturing yield, when the component values follow a Kumaraswamy distribution. Reyad and Ahmed [18] obtained Bayes estimates for the shape parameter α with known λ under the symmetric and asymmetric loss functions. Sindhu et al. [19] investigated the Bayesian and non-Bayesian statistical inferences on the basis of the Type-II censored samples from Kumaraswamy distribution. Moreover, the Bayesian and non-Bayesian statistical inferences based on record values from the Kumaraswamy distribution has been studied by Nadar et al. [20]. Recently, Kohansal and Nadarajah [21] derived the Bayesian and non-Bayesian estimates of the stress-strength parameter under the Type-II hybrid progressive censored samples coming from Kumaraswamy distributions. Also, see Mitnik [22], Wang [23] and Kohansal [24].

The rest of the paper is organized as follows. In Section 2, the ML estimates for the parameters α and β, based on progressively Type-I interval censored data are derived. Section 3 explains how the EM algorithm is used to obtain the ML estimates for the parameters. Ultimately, a simulated censored data set is discussed for illustrative aims in Section 4.

2 Maximum likelihood estimates

Suppose that the progressively Type-I interval censored sample $X_1,\ldots,X_m$ comes from a distribution with CDF F. Then, the likelihood function based on the observed sample is

$$L_p(\boldsymbol{\theta}) \propto \prod_{i=1}^{m}\left[F(t_i)-F(t_{i-1})\right]^{X_i}\left[1-F(t_i)\right]^{R_i}. \tag{3}$$

Under the Kumaraswamy distribution with CDF (2), the likelihood function (3) is reduced to

$$L_p(\boldsymbol{\theta}) \propto \prod_{i=1}^{m}\left[\left(1-t_{i-1}^{\lambda}\right)^{\alpha}-\left(1-t_{i}^{\lambda}\right)^{\alpha}\right]^{X_i}\left[\left(1-t_{i}^{\lambda}\right)^{\alpha}\right]^{R_i}.$$

where $\boldsymbol{\theta} = (\alpha, \lambda)$. The corresponding log-likelihood function is

$$\ln L_p(\boldsymbol{\theta}) = \text{ constant} + \sum_{i=1}^{m} X_i \ln\left[\left(1 - t_{i-1}^{\lambda}\right)^{\alpha} - \left(1 - t_i^{\lambda}\right)^{\alpha}\right] + \alpha \sum_{i=1}^{m} R_i \ln\left(1 - t_i^{\lambda}\right).$$

Taking derivative with respect to parameters α and λ and equating to zero, we have

$$\frac{\partial}{\partial\alpha} \ln L_p(\boldsymbol{\theta}) = \sum_{i=1}^{m} X_i \frac{\left(1 - t_{i-1}^{\lambda}\right)^{\alpha} \ln\left(1 - t_{i-1}^{\lambda}\right) - \left(1 - t_i^{\lambda}\right)^{\alpha} \ln\left(1 - t_i^{\lambda}\right)}{\left(1 - t_{i-1}^{\lambda}\right)^{\alpha} - \left(1 - t_i^{\lambda}\right)^{\alpha}} + \sum_{i=1}^{m} R_i \ln\left(1 - t_i^{\lambda}\right) = 0, \tag{4}$$

and

$$\frac{\partial}{\partial\lambda} \ln L_p(\boldsymbol{\theta}) = \alpha \sum_{i=1}^{m} X_i \frac{t_i^{\lambda}\left(1 - t_i^{\lambda}\right)^{\alpha-1} \ln t_i - t_{i-1}^{\lambda}\left(1 - t_{i-1}^{\lambda}\right)^{\alpha-1} \ln t_{i-1}}{\left(1 - t_{i-1}^{\lambda}\right)^{\alpha} - \left(1 - t_i^{\lambda}\right)^{\alpha}} - \alpha \sum_{i=1}^{m} R_i \frac{t_i^{\lambda} \ln t_i}{\left(1 - t_{i-1}^{\lambda}\right)} = 0. \tag{5}$$

The ML estimates of α and λ are derived by solving (4) and (5) numerically. In next section, we compute the ML estimates of these parameters using the EM algorithm.

3 EM algorithm

Let τ_{ij} with $j = 1, \ldots, X_i$ be the lifetimes of the units failed in the interval $(t_{i-1}, t_i]$ and $\eta_{ij'}$ with $j' = 1, \ldots, R_i$ be the lifetimes of the units censored at the time t_i for $i = 1, \ldots, m$. Then, the likelihood function for the complete sample under the Kumaraswamy distribution with PDF (1) is given by

$$L_c(\theta) \propto \prod_{i=1}^{m} \left(\prod_{j=1}^{X_i} f(\tau_{ij}) \prod_{j'=1}^{R_i} f(\eta_{ij'}) \right) = (\alpha\lambda)^n \prod_{i=1}^{m} \left(\prod_{j=1}^{X_i} \tau_{ij} \prod_{j'=1}^{R_i} \eta_{ij'} \right)^{\lambda-1} \times \prod_{i=1}^{m} \left(\prod_{j=1}^{X_i} (1 - \tau_{ij}^{\lambda}) \prod_{j'=1}^{R_i} (1 - \eta_{ij'}^{\lambda}) \right)^{\alpha-1}.$$

Then, the associated log-likelihood function is obtained as

$$\ln L_c(\theta) = \text{constant} + n \ln\alpha + n \ln\lambda + (\lambda - 1) \sum_{i=1}^{m} \left[\sum_{j=1}^{X_i} \ln \tau_{ij} + \sum_{j'=1}^{R_i} \ln \eta_{ij'} \right] + (\alpha - 1) \sum_{i=1}^{m} \left[\sum_{j=1}^{X_i} \ln\left(1 - \tau_{ij}^{\lambda}\right) + \sum_{j'=1}^{R_i} \ln\left(1 - \eta_{ij'}^{\lambda}\right) \right]. \tag{6}$$

On the basis of the complete sample, the ML estimates of α and λ are obtained by deriving the log-likelihood function in (6) with respect to α and λ and equating to zero. Then, we have

$$\frac{\partial}{\partial \alpha} \ln L_c(\boldsymbol{\theta}) = \frac{n}{\alpha} + \sum_{i=1}^{m} \sum_{j=1}^{X_i} \ln\left(1-\tau_{ij}^{\lambda}\right) + \sum_{i=1}^{m} \sum_{j'=1}^{R_i} \ln\left(1-\eta_{ij'}^{\lambda}\right) = 0, \tag{7}$$

and

$$\begin{aligned} \frac{\partial}{\partial \lambda} \ln L_c(\boldsymbol{\theta}) = & \frac{n}{\lambda} + \sum_{i=1}^{m} \sum_{j=1}^{X_i} \ln \tau_{ij} - (\alpha-1) \sum_{i=1}^{m} \sum_{j=1}^{X_i} \left[\frac{\tau_{ij}^{\lambda} \ln \tau_{ij}}{1-\tau_{ij}^{\lambda}}\right] \\ & + \sum_{i=1}^{m} \sum_{j'=1}^{R_i} \ln \eta_{ij'} - (\alpha-1) \sum_{i=1}^{m} \sum_{j'=1}^{R_i} \left[\frac{\eta_{ij'}^{\lambda} \ln \eta_{ij'}}{1-\eta_{ij'}^{\lambda}}\right] = 0. \end{aligned} \tag{8}$$

For computing the ML estimates of parameters α and λ using the EM algorithm, one requires to perform two steps: expectation step (E-step) and maximization step (M-step). In order to perform E-step, it is required to compute the conditional expectations which are

$$\begin{aligned} A_i(\alpha, \lambda) &= E\left[\ln\left(1-\tau_{ij}^{\lambda}\right) \middle| t_{i-1} < \tau_{ij} \le t_i\right] = \frac{1}{F(t_i) - F(t_{i-1})} \int_{t_{i-1}}^{t_i} \ln\left(1-x^{\lambda}\right) f(x)\,\mathrm{d}x \\ &= \frac{\alpha}{\left(1-t_{i-1}^{\lambda}\right)^{\alpha} - \left(1-t_i^{\lambda}\right)^{\alpha}} \int_{1-t_i^{\lambda}}^{1-t_{i-1}^{\lambda}} u^{\alpha-1} \ln u \; \mathrm{d}u, & (9) \\ B_i(\alpha, \lambda) &= E\left[\ln \tau_{ij} \middle| t_{i-1} < \tau_{ij} \le t_i\right] = \frac{1}{F(t_i) - F(t_{i-1})} \int_{t_{i-1}}^{t_i} \ln x\, f(x)\,\mathrm{d}x \\ &= \frac{\alpha\lambda^{-1}}{\left(1-t_{i-1}^{\lambda}\right)^{\alpha} - \left(1-t_i^{\lambda}\right)^{\alpha}} \int_{1-t_i^{\lambda}}^{1-t_{i-1}^{\lambda}} u^{\alpha-1} \ln(1-u)\,\mathrm{d}u, & (10) \\ C_i(\alpha, \lambda) &= E\left[\frac{\tau_{ij}^{\lambda} \ln \tau_{ij}}{1-\tau_{ij}^{\lambda}} \middle| t_{i-1} < \tau_{ij} \le t_i\right] = \frac{1}{F(t_i) - F(t_{i-1})} \int_{t_{i-1}}^{t_i} \frac{x^{\lambda} \ln x}{1-x^{\lambda}} f(x)\,\mathrm{d}x \\ &= \frac{\alpha\lambda^{-1}}{\left(1-t_{i-1}^{\lambda}\right)^{\alpha} - \left(1-t_i^{\lambda}\right)^{\alpha}} \int_{1-t_i^{\lambda}}^{1-t_{i-1}^{\lambda}} u^{\alpha-2}(1-u) \ln(1-u)\,\mathrm{d}u, & (11) \\ A_i^*(\alpha, \lambda) &= E\left[\ln\left(1-\eta_{ij'}^{\lambda}\right) \middle| \eta_{ij'} \ge t_i\right] = \frac{1}{1-F(t_i)} \int_{t_i}^{1} \ln\left(1-x^{\lambda}\right) f(x)\,\mathrm{d}x \\ &= \frac{\alpha}{\left(1-t_i^{\lambda}\right)^{\alpha}} \int_{0}^{1-t_i^{\lambda}} u^{\alpha-1} \ln u \; \mathrm{d}u, & (12) \\ B_i^*(\alpha, \lambda) &= E\left[\ln \eta_{ij'} \middle| \eta_{ij'} \ge t_i\right] = \frac{1}{1-F(t_i)} \int_{t_i}^{1} \ln x\, f(x)\,\mathrm{d}x \\ &= \frac{\alpha\lambda^{-1}}{\left(1-t_i^{\lambda}\right)^{\alpha}} \int_{0}^{1-t_i^{\lambda}} u^{\alpha-1} \ln(1-u)\,\mathrm{d}u, & (13) \end{aligned}$$

and

$$C_i^*(\alpha,\lambda) = E\left[\frac{\eta_{ij'}^{\lambda}\ln\eta_{ij'}}{1-\eta_{ij'}^{\lambda}}\Big|\,\eta_{ij'}\geq t_i\right] = \frac{1}{1-F\left(t_i\right)}\int_{t_i}^{1}\frac{x^{\lambda}\ln x}{1-x^{\lambda}}\,f(x)\,\mathrm{d}x$$
$$= \frac{\alpha\lambda^{-1}}{\left(1-t_i^{\lambda}\right)^{\alpha}}\int_0^{1-t_i^{\lambda}} u^{\alpha-2}(1-u)\ln(1-u)\,\mathrm{d}u. \qquad (14)$$

Using (1) to (14) and taking expectations of both sides of (7) and (8), we have

$$\frac{n}{\alpha}+\sum_{i=1}^{m}\left[X_iA_i(\alpha,\lambda)+R_iA_i^*(\alpha,\lambda)\right]=0, \qquad (15)$$

and

$$\frac{n}{\lambda}+\sum_{i=1}^{m}\left[X_iB_i(\alpha,\lambda)+R_iB_i^*(\alpha,\lambda)\right]-(\alpha-1)\sum_{i=1}^{m}\left[X_iC_i(\alpha,\lambda)+R_iC_i^*(\alpha,\lambda)\right]=0. \qquad (16)$$

The M-step deals with maximizing (15) and (16) with respect to the parameters α and λ. Hence, if the estimates of α and λ at the k-th stage of the iteration are $\hat{\alpha}_{(k)}$ and $\hat{\lambda}_{(k)}$, the updated estimates of α and λ at the $(k+1)$-th stage are computed by the formulas

$$\hat{\alpha}_{(k+1)} = -\,n\left(\sum_{i=1}^{m}\left[X_iA_i\big(\hat{\alpha}_{(k)},\hat{\lambda}_{(k)}\big)+R_iA_i^*\big(\hat{\alpha}_{(k)},\hat{\lambda}_{(k)}\big)\right]\right)^{-1},$$

and

$$\hat{\lambda}_{(k+1)} = n\Bigg(\big(\hat{\alpha}_{(k+1)}-1\big)\sum_{i=1}^{m}\left[X_iC_i\big(\hat{\alpha}_{(k+1)},\hat{\lambda}_{(k)}\big)+R_iC_i^*\big(\hat{\alpha}_{(k+1)},\hat{\lambda}_{(k)}\big)\right]$$
$$-\sum_{i=1}^{m}\left[X_iB_i\big(\hat{\alpha}_{(k+1)},\hat{\lambda}_{(k)}\big)+R_iB_i^*\big(\hat{\alpha}_{(k+1)},\hat{\lambda}_{(k)}\big)\right]\Bigg)^{-1}.$$

The EM algorithm is repeated until the desired convergence is satisfied. The desired convergence is difined as $|\hat{\alpha}_{(k+1)}-\hat{\alpha}_{(k)}|+|\hat{\lambda}_{(k+1)}-\hat{\lambda}_{(k)}|<\epsilon$ for a small value of $\epsilon>0$. It is guaranteed that the EM algorithm will always converge to a local maximum of the likelihood function. For more details, see, for example, Rud [25], Jordan [26] and Wu [27]. The reasonable initial values for $\hat{\alpha}_{(0)}$ and $\hat{\lambda}_{(0)}$ are the ML estimates of these parameters based on the pseudo-complet sample by replacing the observations τ_{ij} $(j=1,\ldots,X_i)$ and $\eta_{ij'}$ $(j'=1,\ldots,R_i)$ by t_i for $i=1,\ldots,m$.

4 Illustrative example

The objective of this section is to analyse a simulated data set. We use the following algorithm to simulate the numbers of failures observed X_i in intervals $(t_{i-1},t_i]$ for $i=1,\ldots,m$, from an initial sample of size n placed on a life test at time $t_0=0$.

Assuming $X_0 = 0$ and $R_0 = 0$, we have

$$X_i|\left(X_{i-1}, R_{i-1}, \ldots, X_i, R_i\right) \sim \text{Binomial}\left(n - \sum_{j=1}^{i-1}\left(X_j + R_j\right), q_i\right),$$

for $i = 1, \ldots, m$, where

$$q_i = \frac{F(t_i) - F(t_{i-1})}{1 - F(t_{i-1})}.$$

Then, a progressively Type-I interval censored sample is generated using the following algorithm:

- Let Let $N = 0$ and $R = 0$.
- For i from 1 to m do

   1. generate X_i from the binomial distribution with parameters $n - N - R$ and q_i,
   2. compute $R_i^* = \lfloor p_i\left(n - N - R - X_i\right)\rfloor$ or $R_i^* = \min\left(R_i, n - N - R - X_i\right)$,
   3. let $N = N + X_i$ an $R = R + R_i^*$,
   4. if $i < m$ and $n - N - R \neq 0$, go to Step (1); otherwise, stop.

For simulating a progressively Type-I interval cencored sample, we considered $n = 30$ and $m = 5$ with inspection times $t_1 = 0.1$, $t_2 = 0.2$, $t_3 = 0.4$, $t_4 = 0.6$ and $t_5 = 0.8$, and censoring scheme $(p_1, \ldots, p_m) = (0.25, 0.25, 0.5, 0.5, 1)$. Also, we supposed $\alpha = 3$ and $\lambda = 2$. Then, the simulated values are $(X_1, \ldots, X_m) = (2, 2, 4, 5, 0)$ and $(R_1^*, \ldots, R_m^*) = (7, 4, 5, 0, 1)$. Using this censored sample and letting $\epsilon = 0.001$, the ML estimates for α and λ via the EM algorithm need 42 iterations to converge to 2.1814 and 1.5755, respectively.

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Testing stochastically increasing by non-parametric kernel method

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Abstract

This paper, investigates the problem of testing that non-negative random variables X and Y are independent against the alternative that Y stochastically increasing (SI) in X. Two new statistical tests, based on the kernel density estimator, are proposed. Their limiting distributions are derived. The finite-sample performance of the proposed tests in comparison with various alternative tests, is studied.

Keywords: Stochastically increasing, Kernel estimation, Empirical processes, Wiener process, Brownian bridge process, Copula function.

1 Introduction

Establishing dependence relations between random variables is one of the most widely studied subjects in probability and statistics and have shown to be very useful tool in insurance, actuarial science and various aspects of reliability. Many studies in statistics are designed to explore the relationship between random variables X and Y, say, and specially to determine whether X and Y are independent or dependent. The simplest way to obtain the dependence structure between two random variables may be calculating the value of their covariance. However, the measure of covariance is usually used to obtain the linear dependency relationship between

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them. Therefore, it has been defined many notions of positive dependences. We may have seen abbreviations like these *totally positive of order 2* (TP_2), *stochastically increasing* (SI), *right tail increasing* (RTI), and so on. For discussions and applications on the various notions of dependence, we refer to Lehmann (1966), Esary and Proschan (1972), Harris (1970) and Block and Ting (1981) and for a comprehensive review of concept of dependence the reader is referred to Lai and Xie (2006).

In the reliability literature, the components of a system are assumed to be statistically independent. However, in practices, the components of a system might be dependent because of the environmental factors or the shared load, and hence the failure of one affects the performance of the other components. Therefore, an engineer may be interested in the dependence relationship between components of the system reliability and the environment factor. Therefore, the concept of stochastic dependence between two or more random variables play an important part in statistics. Various notions of dependence are motivated from application in statistical reliability (cf, Barlow and Proschan (1981)).

Let (X, Y) be an absolutely continuous bivariate random vector with marginal distribution functions F and G respectively, and joint distribution function $H(x, y)$, $(x, y) \in \mathcal{R}^+ \times \mathcal{R}^+$, where $\mathcal{R}^+ = (0, \infty)$. The notion of SI dependence between random variables X and Y is denoted by $SI(Y|X)$, which is introduced as follows.

Definition 1.1. Random variable Y is said to be stochastically increasing in x, $x > 0$ for all $y \in \mathcal{R}^+$, if, for every $y \in \mathcal{R}^+$, $P(Y > y|X = x)$ is increasing in x.

2 The test statistics

Let $H(x, y)$ be the joint probability distribution function of an absolutely continuous random vector (X, Y) defined on some probability space $(\Omega, \mathcal{R}^2, \mathcal{P})$ where (X, Y) takes values on $\mathcal{R}^+ \times \mathcal{R}^+$, and $F(x)$ and $G(y)$ be corresponding marginal distribution functions. If $SI(Y|X)$ hold then

$$\mathbb{P}(Y > y|X = s) \geq \mathbb{P}(Y > y|X = t); \text{ whenever } \; s \geq t > 0.$$

For any fixed $y > 0$, $\delta(s, t; y)$ is defined by

$$\delta(s, t; y) = \mathbb{P}(Y > y|X = s) - \mathbb{P}(Y > y|X = t); \text{ whenever } s \geq t > 0. \tag{1}$$

Based on it, we define the measure of deviation from H_0 (hypothesis that X and Y are independent) to H_1 (hypothesis thata $SI(Y|X)$ hold) by

$$\Delta_{(F,G)}(y) = \mathsf{E}[\delta(X_2, X_1; y)|X_2 \geq X_1].$$

We have that, for any $y > 0$,

$$
\begin{aligned}
\Delta_{(F,G)}(y) &= \int\int_{s\geq t} \delta(s,t)dF(t)dF(s) \\
&= \int\int_{s\geq t} \overline{G}(y|s)dF(s)dF(t) - \int\int_{s\geq t} \overline{G}(y|t)dF(t)dF(s) \\
&= \int \overline{G}(y|s)F(s)dF(s) - \int \overline{G}(y|t)\overline{F}(t)dF(t) \\
&= \int \overline{G}(y|x)F(x)dF(x) - \int \overline{G}(y|x)\overline{F}(x)dF(x) \\
&= \int \overline{G}(y|x)(2F(x)-1)dF(x), \qquad (2)
\end{aligned}
$$

Notice that $\Delta_{(F,G)}(y) = 0$ under H_0, and $\Delta_{(F,G)}(y) > 0$ under H_1.
Hence, we propose tests based on the following two measures

$$\Delta^*(F,G) = \sup_{y\in\mathcal{R}^+} \Delta_{(F,G)}(y). \qquad (3)$$

and

$$\Delta^{**}(F,G) = \int \Delta_{(F,G)}(y)dG(y). \qquad (4)$$

Now, based on a sample of independent identically distributed random vectors (X_1, Y_1), (X_2, Y_2) ,..., (X_n, Y_n) with distribution function $H(x, y)$, we construct the test statistics for testing H_0 against H_1. To estimate the conditional survival distribution function, we use the Nadaraya-Watson approach proposed by Nadaraya (1964) and Watson (1964). More details about it can be found in Simonoff (1996), Li and Racine (2007), Hall et al. (1999) and Cai (2002). It is given by

$$\overline{G}_n(y|x) = \frac{\sum_{j=1}^n k(\frac{x-X_j}{a_n})I(Y_j > y)}{\sum_{j=1}^n k(\frac{x-X_j}{a_n})}, \qquad (5)$$

where a_n is a is a sequence of positive real numbers, which are often referred to as bandwidth in the literature (see Silverman (1986) for more details), and k is so called kernel function. Silverman (1986) for more details). The kernel density estimator of $f(x)$, $f_n(x) = \frac{1}{na_n}\sum_{j=1}^n k(\frac{x-X_j}{a_n})$, appeared in the denominator (5).

A natural estimator of $\Delta_{(F,G)}(y)$ is now given by

$$\Delta_{(F_n,G_n)}(y) = \int \overline{G}_n(y|x)(2F_n(x)-1)dF_n(x),$$

where $F_n(x) = \frac{1}{n}\sum_{i=1}^n I(X_i \leq x)$ is empirical distribution function. Now we have all ingredients to construct test statistics based on the measures of departure (3) and (4), We propose the supremum-type statistic

$$\Delta^*{}_{(F_n,G_n)} = \sup_{y\in\mathcal{R}^+} \Delta_{(F_n,G_n)}(y),$$

and the integral-type statistic

$$\Delta^{**}{}_{(F_n,G_n)} = \int \Delta_{(F_n,G_n)}(y)dG_n(y). \tag{6}$$

From (5) and (2), we have

$$\Delta_{(F_n,G_n)}(y) = \frac{1}{n^2}\sum_{i,j,l} \frac{k(\frac{X_i-X_j}{a_n})I(Y_j > y)(2I(X_i \geq X_l)-1)}{\sum_{j=1}^n k(\frac{X_i-X_j}{a_n})}. \tag{7}$$

Hence, the statistic (6) can be expressed in following form, much more suitable for computation.

$$\begin{aligned}
\Delta^{**}_{(F_n,G_n)} &= \int \frac{1}{n^2}\sum_{i,j,l} \frac{k(\frac{X_i-X_j}{a_n})I(Y_j > y)(2I(X_i \geq X_l)-1)}{\sum_{j=1}^n k(\frac{X_i-X_j}{a_n})} dG_n(y) \\
&= \frac{1}{n^3}\sum_{i,j,l} \frac{k(\frac{X_i-X_j}{a_n})\sum_{k=1}^n I(Y_j > Y_k)(2I(X_i \geq X_l)-1)}{\sum_{j=1}^n k(\frac{X_i-X_j}{a_n})} \\
&= \frac{1}{n^3}\sum_{i,j,l} \frac{k(\frac{X_i-X_j}{a_n})\left[\sum_{k=1}^n I(Y_j \geq Y_k)-1\right](2I(X_i \geq X_l)-1)}{\sum_{j=1}^n k(\frac{X_i-X_j}{a_n})} \\
&= \frac{1}{n^3}\sum_{i,j} \frac{k(\frac{X_i-X_j}{a_n})(S_j-1)(2R_i-n)}{\sum_{j=1}^n k(\frac{X_i-X_j}{a_n})}
\end{aligned}$$

where $R_i = \text{Rank}(X_i) = \sum_{l=1}^n I(X_i \geq X_l)$ and $S_j = \text{Rank}(Y_j) = \sum_{k=1}^n I(Y_j \geq Y_k)$.

Naturally, we take large values of test statistics to we significant. As a consequence, both tests are consistent against considered class of alternatives.

3 Asymptotic distribution of the test statistics

In this section, we examine asymptotic properties of our test statistics. We use the notation "$\Rightarrow$" to indicate convergence in distribution. Before we state main result, we introduce some assumptions.

Assumption A:

- Let $h(x,y)$ be the density function corresponding to $H(x,y)$. Suppose that $h(x,y)$ has the bounded first and the second partial derivative with respect to x, which are denoted by $h^{(1)}(x,y)$ and $h^{(2)}(x,y)$, respectively.
- Let the conditional survival function of Y given $X = x$ be $\overline{G}(y|x)$, and $g(y|x)$ be the corresponding conditional density function of Y given $X = x$.
- Let $m(x,y) = \int I(u \leq y)h(x,u)du$, be uniformly bounded up until the second derivative.
- Let the marginal density functions of X and Y, denoted by $f(x)$ and $g(y)$, respectively are uniformly continuous and bounded up to the second derivative.

Assumption B:

Suppose, we have a symmetric and bounded kernel density function $k : \mathcal{R} \to \mathcal{S} \subset \mathcal{R}$ that satisfies:

- $\int k(s)ds = 0$ and $\int k(^2 s)ds < \infty$;
- $\sup |k(s)| < \infty$, and $\int |k(s)|ds < \infty$,and $\int [k(s)]^{2+\delta} ds < \infty$, for some $\delta > 0$;
- $\int |k'(s)|ds < \infty$.

Assumption C:

Suppose that a_n be a sequence of positive real numbers that satisfy the following conditions:

- $a_n \to 0$, as, $n \to \infty$.
- $na_n^2 \to \infty$, as, $n \to \infty$.
- $na_n^5 \to 0$, as, $n \to \infty$.

The following theorem gives us the asymptotic behavior of the empirical process $\{\Delta_{(F_n,G_n)}(y),\ y > 0\}$.

Theorem 3.1. *Under assumptions* **A**, **B** *and* **C**, *it holds that*

$$\sqrt{na_n}\left(\Delta_{(F_n,G_n)}(y) - \Delta_{(F,G)}(y)\right) \Rightarrow \mathcal{Q}(y);\ \textit{whenever } y \in \mathcal{R}^+, \tag{8}$$

in $D(0,\infty)$, *where,* $\mathcal{Q}(y) = k_0^{\frac{1}{2}} \int [f(x)]^{-\frac{1}{2}}(2F(x)-1)\mathcal{B}(G(y|x))dF(x)$, $k_0 = \int k^2(s)ds$ *and* $\mathcal{B}$ *is a standard Brownian Bridge process on the unit interval* $[0,1]$.

As a corollary, we get the asymptotic distributions of test statistics.

Theorem 3.2. *Under assumptions* **A**, **B** *and* **C**, *it holds that*

$$\sqrt{na_n}\left(\Delta^*_{(F_n,G_n)} - \Delta^*_{(F,G)}\right) \Rightarrow \sup_{y\in\mathcal{R}^+} \mathcal{Q}(y),$$

where,

$$\sup_{y\in\mathcal{R}^+} \mathcal{Q}(y) = k_0^{\frac{1}{2}} \sup_{y\in\mathcal{R}^+} \int [f(x)]^{-\frac{1}{2}}(2F(x)-1)\mathcal{B}(G(y|x))dF(x).$$

Notice that, under the null hypothesis i.e. X and Y are independent, $\sup_{y\in\mathcal{R}^+} \mathcal{Q}(y)$ can be rewritten as follows

$$\begin{aligned}\sup_{y\in\mathcal{R}^+} \mathcal{Q}(y) &= k_0^{\frac{1}{2}} \sup_{y\in\mathcal{R}^+} \int [f(x)]^{-\frac{1}{2}}(2F(x)-1)\mathcal{B}(G(y))dF(x)\\ &= k_0^{\frac{1}{2}} \sup_{y\in\mathcal{R}^+} \mathcal{B}(G(y)) \int [f(x)]^{-\frac{1}{2}}(2F(x)-1)dF(x).\end{aligned}$$

Next, we establish the asymptotic distribution of statistic $\Delta^{**}_{(F_n,G_n)}$ under the null hypothesis.

Theorem 3.3. *Under assumptions* **A**, **B**, **C** *and the null hypothesis, it holds that*

$$k_0^{-\frac{1}{2}}\Gamma^{-1}\sqrt{na_n}\left(\Delta^{**}_{(F_n,G_n)} - \Delta^{**}_{(F,G)}\right) \Rightarrow N\big(0, \frac{1}{12}\big), \quad as,\ n \to 0.$$

where $\Gamma = \int [f(x)]^{-\frac{1}{2}}(2F(x)-1)dF(x)$.

Proof. From Theorem 3.1 we have

$$\begin{aligned}\sqrt{na_n}\left(\Delta^{**}_{(F_n,G_n)} - \Delta^{**}_{(F,G)}\right) &\Rightarrow \int \mathcal{Q}(y)dG(y) \\ &= k_0^{\frac{1}{2}}\int [f(x)]^{-\frac{1}{2}}(2F(x)-1)dF(x)\int \mathcal{B}(G(y))dG(y) \\ &= k_0^{\frac{1}{2}}\Gamma\int_0^1 \mathcal{B}(u)du.\end{aligned}$$

It can be easily shown that $\int_0^1 \mathcal{B}(u)du$ is centred normal random variable with variance $\frac{1}{12}$. $\square$

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Machine learning methods to predict the deaths of people experiencing shocks

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Abstract

One of the factors that is influential in mortality rates and can change this rate, is shock which people can experience in their lives. With consideration of a stochastic process, non-homogeneous poisson process, some of these mortality rates with shocks have defined and we aim to use these processes and their information and reliability to predict the number of people who are influenced by this process with a specific rate, at a special age; So we use some of machine learning methods for our purpose and find the best one of them.

Keywords: Mortality rate, Shock, Stochastic process, Reliability, Machine learning.

1 Introduction

There are various stochastic modeling of aging of organisms in the literature. For example, in vitality models (Li and Anderson 2009), deterioration is modeled by a stochastic process (e.g., by a Wiener process with negative drift), describing the non-monotone decrease in vitality of organisms. However in another consideration, we employ a different reasoning and instead of

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dealing with direct deterioration of vital parameters, we consider mortality rate as an increasing stochastic process and describe stochastic aging of organisms. We will consider a specific mortality rate process governed by the non-homogeneous Poisson process of point events (shocks) and will show that, although the sample paths of the unconditional mortality rate process are monotonically increasing, the population mortality rate can decrease with age. There are some real examples of these decreasing mortality rates such as Carey et al. (1992) for the population of medflies and recent analysis of human populations at advanced ages exhibit deceleration and leveling off the Gompertz mortality curve at advanced ages (Kannisto et al. 1994; Missov and Finkelstein 2011). Thus this introduced model enables to analyze and interpret the shape of the resulting population mortality rate and its deceleration with age.

For the first time the idea of using shocks modeling for describing patterns of mortality rates from the famous paper by Strehler and Mildvan (1960) was introduced where the organism is exposed to the Poisson process of shocks that are interpreted as demands for energy, during illnesses or disorders, whereas death occurs when the magnitude of this demand exceeds the current vitality of an organism. So with using this model, the rate of mortality in Gompertz model that was always with this increased pattern, could be modified.

To consider heterogeneity in mortality rates rather than just increasing models for all of the organisms, some models have introduced with papers by Beard (1959) and Vaupel et al. (1979), and they used a multiplicative gamma-Gompertz model was defined as the following mortality rate process

$$\mu_t = Zae^{bt} \tag{1}$$

The random variable Z is often called frailty and it describes the fixed heterogeneity of population assigned at t = 0 and not changing in time. However for having a modified model we consider the model which will introduce in the next section that is known as a evolving heterogeneity and it is not only different for one person to another one but also various during every person life for their different experiences.

We use the information and reliability function of these models and consider some simulations for time of deaths based on these models; then based on various machine learning methods we will predict death for a new person in a specific age, while we know they have a special rate of this stochastic processes and shocks.

2 Introduction of the model

While there are multiplicative and additive types for hazard rates, with considering an additive model, a specific mortality rate is introduced as the following

$$\mu_t = \mu_0(t) + \eta N(t) \tag{2}$$

and with conditioning on survivors, another model can describe that is useful for deceleration mortality rates with age

$$\{\mu_t | T > t\} = \mu_0(t) + \eta\{N(t)|T > t\} \tag{3}$$

Where $\mu_0(t)$ is a background mortality rate that models the common environment for organisms. Therefore, crucial term will be stochastic $\eta N(t)$.

η is a deterministic jump on each event from the point process. Thus the damage incurred by organism from a shock is translated into a jump in the corresponding mortality rate. In other words, when each shock happens, we have η unit increasing on the mortality rate.

$\{N(t), t \geq 0\}$ is a point stochastic process. We will assume that it is the nonhomogeneous Poisson process (NHPP) with rate $\lambda(t)$. For convenience, and in line with some existing models and based on some papers (Strehler and Mildvan 1960; Finkelstein 2008; Cha and Mi 2007), we will use the term shocks for events from this process.

With expectation of the (2), we will have

$$E(\mu_t) = \mu_0(t) + \eta E(N(t))$$

and as we know, the $N(t)$ is a non-homogeneous Poisson process so

$$E(N(t)) = \int_0^t \lambda(x)dx$$

and therefore

$$E(\mu_t) = \mu_0(t) + \eta \int_0^t \lambda(x)dx. \quad (4)$$

Applying the operation of mathematical expectation with respect to both sides of (2) results in the above expression for the population mortality rate.

In the next section we will use the information of this model to simulate the times and considering machine learning methods on them.

3 Simulation and using machine learning methods

To simulate the time of deaths for some people who are influenced by shocks based on the (2) model, with knowing the laws and relationships between Cumulative distribution function, Reliability function and Hazard rate, we have (u is the random sample from the Uniform distribution)

$$F(t) = 1 - R(t) = 1 - exp(-\int_0^t h(x)dx) = u \quad (5)$$

In this general equation $h(x)$ is the hazard rate and in our work this rate is mortality rate and in other words, it is $E(\mu_t)$. So based on (2) and (5) we have

$$exp(-\int_0^t (\mu_0(t) + \eta \int_0^t \lambda(u)du)\mathrm{d}x) = 1 - u = U \quad (6)$$

Now with considering different $\lambda(x)$ that is the rate of our stochastic process, we will reach the time of deaths for people based on that process.(Who are expriencing shocks based on that specific process)

Example 3.1. With having the homogeneous Poisson process and $\lambda(x) = \lambda$, $\mu_0(t) = 0$ and $\eta = 1$, based on the (6) we have

$$exp(-\int_0^t \lambda x dx) = U \tag{7}$$

And then with $\lambda(x) = \lambda = 1$ and solving the above equation

$$exp(-\frac{t^2}{2}) = U \tag{8}$$

Now based on the above equation and solving it we have the time of death for 100 sample of individuals. At the first step we have the plot of number of these people in every age, in other words how many people have died in a special age.(For each age, 10 times repetition has done and the mean of them has calculated).

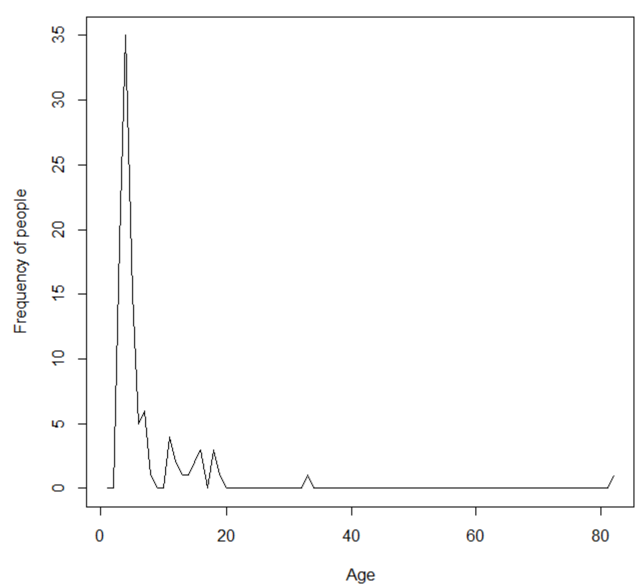


Figure 1: Frequency of people under shocks based on NHPP with $\lambda(x) = \lambda = 1$

To predict the number of deaths in a new age, we use various machine learning methods to predict it. We consider some main methods for this aim

Smoothing splines. In fitting a smooth curve to a set of data, what we really want to do is find some function, say $g(x)$, that fits the observed data well: that is, we want $RSS = \sum_{i=1}^{n}(y_i - g(x_i))^2$ to be small. However, there is a problem with this approach. If we dont put any constraints on $g(x_i)$, then we can always make RSS zero simply by choosing g such that it interpolates all of the y_i. Such a function would woefully overfit the datait would be far too flexible. What we really want is a function g that makes RSS small, but that is also smooth.

How might we ensure that g is smooth? There are a number of ways to do this. A natural approach is to find the function g that minimizes

$$\sum_{i=1}^{n}(y_i - g(x_i))^2 + \lambda \int g''(t)^2 dt \tag{9}$$

where λ is a non-negative tuning parameter. The function g that minimizes (9) is known as a smoothing spline.

Optimal spline. In this method we use optimum smoothing splines with using N-fold cross-validation(that is a resampling method that uses different portions of the data to test and train a model on different iterations) and in another way with generalized cross-validation.

Local Regression(LOESS). This method is a different approach for fitting flexible non-linear functions, which involves computing the fit at a target point x_0 using only the nearby training observations. There is an algorithm for this method

Algorithm. Local Regression At $X = x_0$

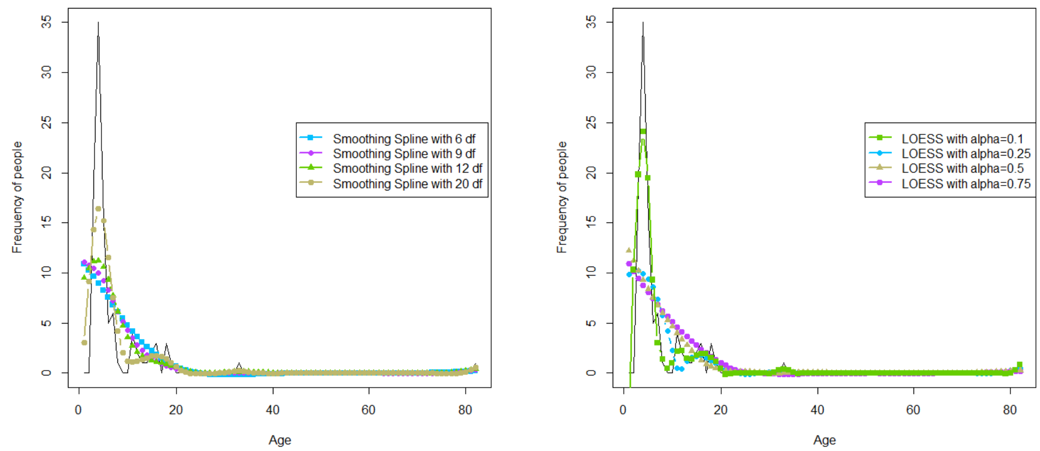
1. Gather the fraction s = k/n of training points whose x_i are closest to x_0.
2. Assign a weight $K_{i0} = K(x_i, x_0)$ to each point in this neighborhood, so that the point furthest from x_0 has weight zero, and the closest has the highest weight. All but these k nearest neighbors get weight zero.
3. Fit a weighted least squares regression of the y_i on the x_i using the aforementioned weights, by finding $\hat{\beta}_0$ and $\hat{\beta}_1$ that minimize

$$\sum_{i=1}^{n} K_{i0}(y_i - \beta_0 - \beta_1 x_i)^2 \tag{10}$$

4. The fitted value at x_0 is given by $\hat{f}(x_0) = \hat{\beta}_0 + \hat{\beta}_1 x_0$.

In order to perform local regression, there are a number of choices to be made, such as how to define the weighting function K, and whether to fit a linear, constant, or quadratic regression in Step 3. While all of these choices make some difference, the most important choice is the span s, which is the proportion of points used to compute the local regression at x_0, as defined in Step 1 above. The span plays a role like that of the tuning parameter λ in smoothing splines: it controls the flexibility of the non-linear fit.

Now we use these methods on results and plot of Example 3.1.



(a) Smothing splines (b) Local regression

Figure 2: Machine learning methods when $\lambda(x) = \lambda = 1$

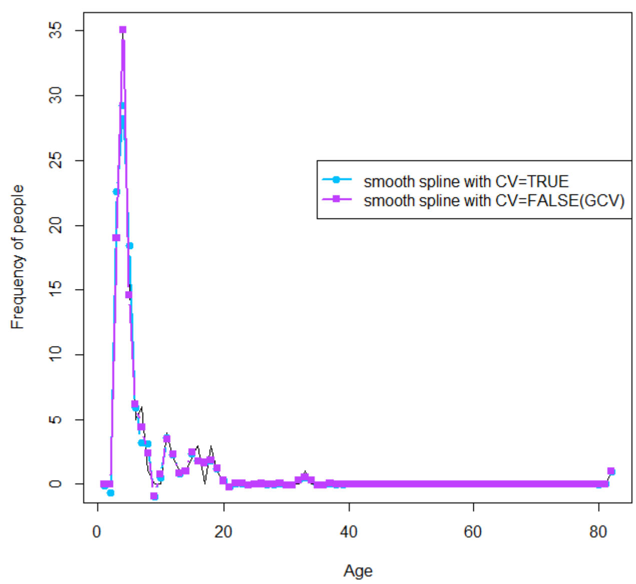


Figure 3: Optimal splines when $\lambda(x) = \lambda = 1$

Example 3.2. Suppose $\lambda(t) = \lambda t$ and then we have a non-homogeneous Poisson process. with $\mu_0(t) = 0$, $\eta = 1$ and $\lambda = 1$, so based on the (6) we have

$$exp(-\int_0^t \frac{1}{2}\lambda t^2 dx) = exp(-\frac{t^3}{6}) = U \tag{11}$$

And based on solving this equation and using machine learning methods we have these plots. (figure (4))

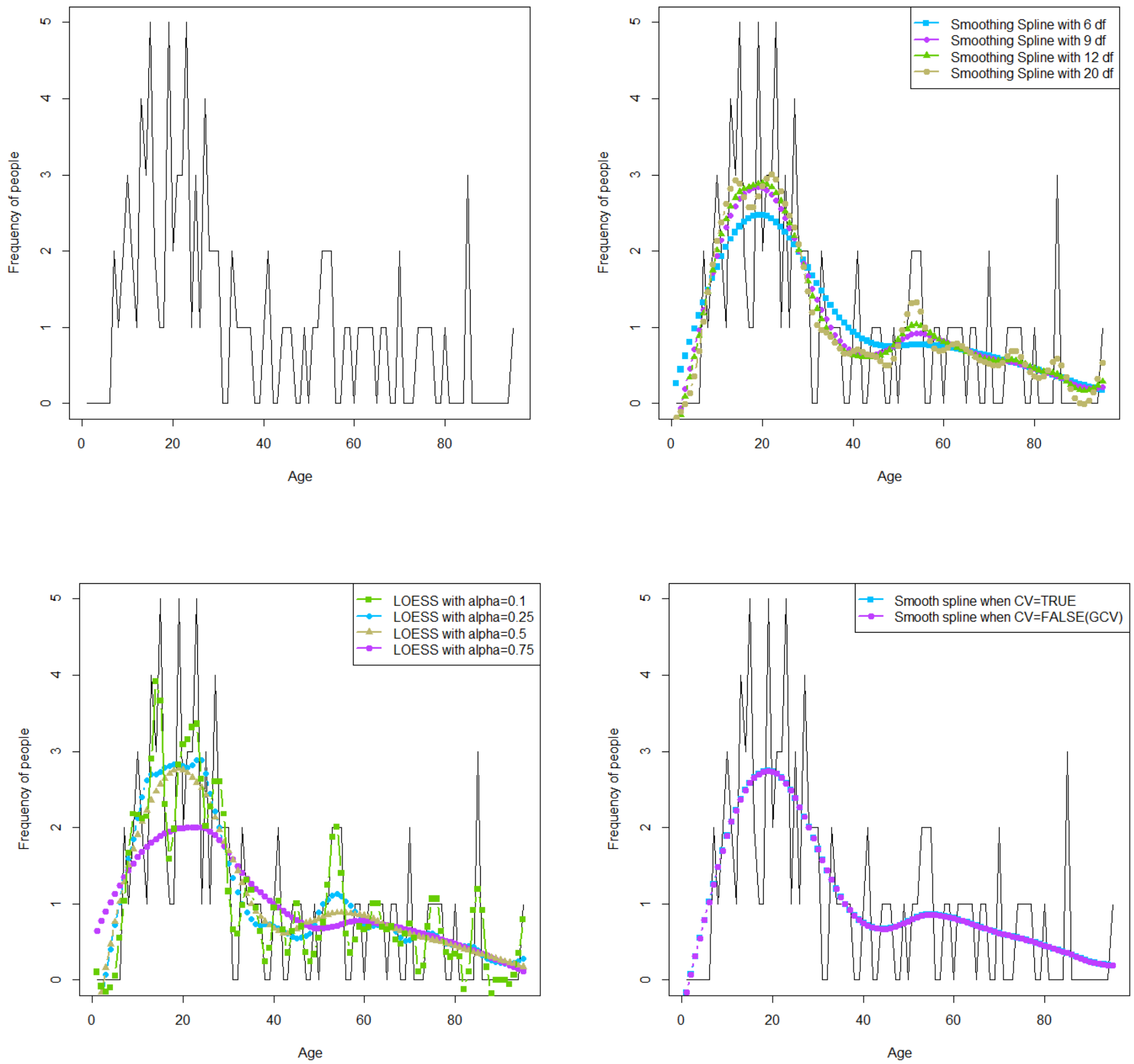


Figure 4: Comparing machine learning methods when the rate of NHPP is $\lambda(t) = \lambda t$

Example 3.3. In the paper of Cha and Finkelstein (2016) we can consider our introduced mortality rate equal to the general Gompertz model and based on this equation we will reach the suitable λ. In other words, we have

$$\mu_0(t) + \eta\, E[N(t)] = a\, exp\{bt\} = a + a(exp\{bt\} - 1) \tag{12}$$

And then the rate of our process will be

$$E(N(t)) = a\,\eta^{-1}(exp\{bt\} - 1) \quad \rightarrow \quad \lambda(t) = ab\eta^{-1}exp\{bt\}$$

With considering $a = 0.1$, $b = 2$ and $\eta = 1$ we must solve the following equation and then the frequency of people who are died based on this specific process is based on the figure (1) and we can see our various machine learning methods on them.

$$exp(-0.1((exp(\frac{2t}{2})) - (\frac{1}{2}))) = U \tag{13}$$

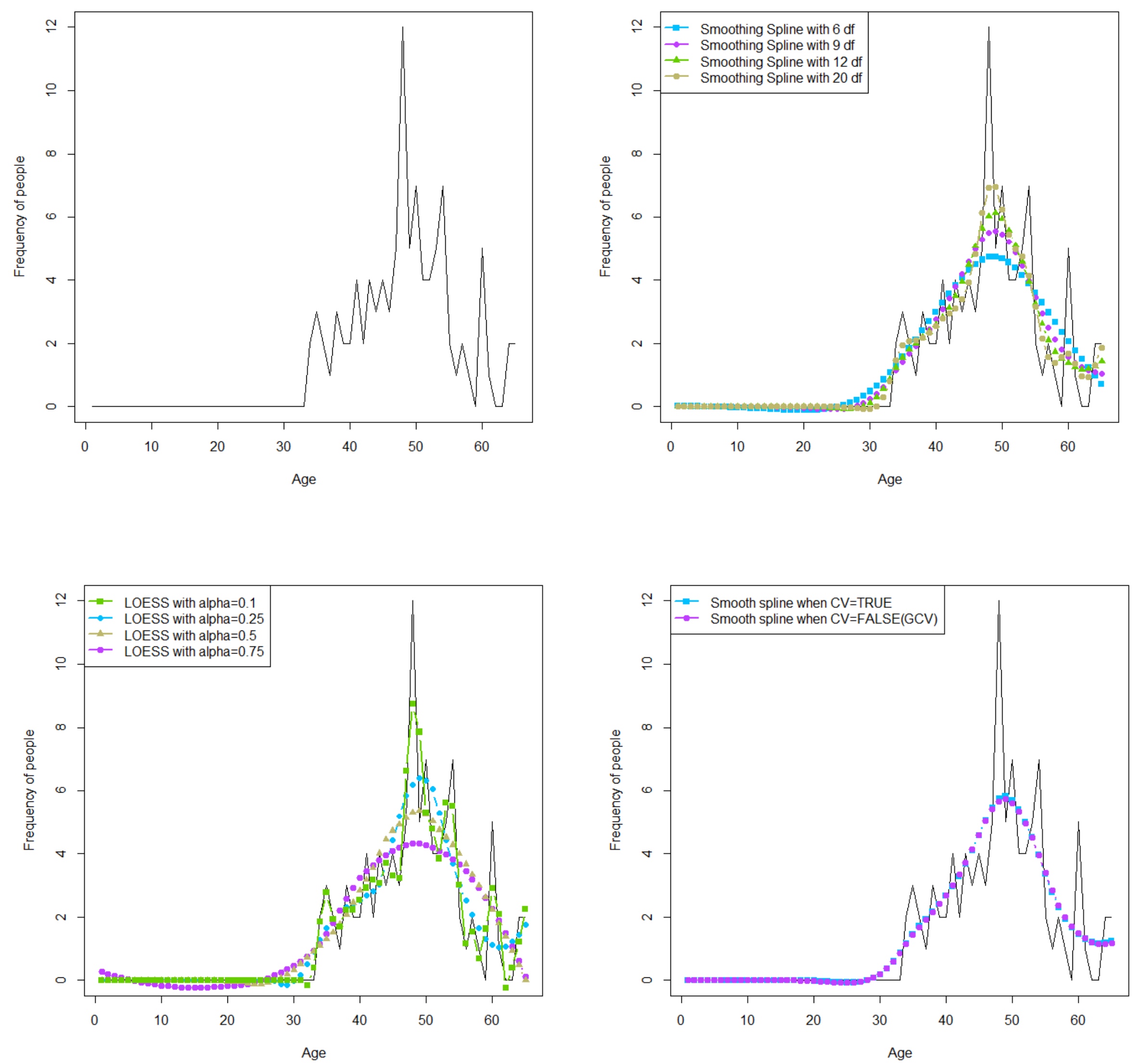


Figure 5: Comparing machine learning methods with considering Gompertz model when the rate of process is $\lambda(t) = ab\eta^{-1}exp\{bt\}$

4 Main results

In this paper, we consider a model for mortality rate that solves some of problems in the classic Gompertz model; with considering shocks that people expreincing and non-homogeneous poisson process to show these shocks we have an evolving heterogenity on the population rather than just increasing model for all of the people in the Gompertz model. The usefulness of this model is in this model every person has a mortality rate of their own and it may be completely different from others.

After introducing this additive model, based on some of equations and with simulation we simulated the time of deaths for people in three processes based on our introduced model and used different machine learning methods to predict the number of people who will die in a specific age. However, in some of these examples the optimal splines have a good results, but consideration of just Local regression method can be enough for us and even the $\lambda = 0.1$ is suitable rather than rates smaller than it and having a more flexible method. Therefore, with having this method when we want predict the number of people died in an age that we have not it, we can use this method to predict it.

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Factors affecting the growth disorder of children under two years of age using multilevel models

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Abstract

Delay or stop in children's growth is referred to as failure to thrive (abbreviated as FTT) which leads to adverse effects such as increased mortality, reduced learning, cognitive, physical, and emotional disability, and other related illnesses. To date, different studies have been carried out in this field and factors affecting growth failure have been identified. Stopping breast feeding, teething, urinary and respiratory tract infection, fever, diarrhea, and malnutrition are identified as the most important factors affecting failure to thrive. Most of these studies apply common regression models; however, multilevel regression models involve the random effects model which allows taking genetic and individual factors into account. In the present study, given that the data were longitudinal and multilevel regression models were used for data analysis, the individual characteristics of children were identified as being among the factors affecting failure to thrive. Accordingly, it can be argued that, in identical conditions, children develop different levels of growth disorder.

Keywords: Multilevel Model, Regression Model, Longitudinal Data, Growth Disorder.

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1 Introduction

Delay or stop in children's growth, dubbed failure to thrive (abbreviated as FTT), is sometimes observed among children under five years of age and leads to adverse effects such as increased mortality, reduced learning, cognitive, physical, and emotional disability, and other related illnesses [1]. To put it differently, FTT can be defined as slower-than-expected physical growth [2]. However, it is worth mentioning that all underweight children are not necessarily afflicted with FTT [3]. Reasons leading to the development of growth failure can be subsumed under two major categories, namely organic and nonorganic reasons. Organic reasons include factors such as suffering from acute or chronic disease (such as infection) interfering with the absorption, metabolism, nutrient intake or increasing energy requirements (such as asthma). The nonorganic reasons, on the other hand, are mostly concerned with environmental, mental, and social factors such as not receiving enough food. In 80% of cases, children are afflicted with growth failure due to nonorganic factors such as impoverishment, decreased appetite, being born in large and extended families, insufficient mother milk (due to undernourishment, psychological stress, etc.), mother's insufficient and poor knowledge regarding correct feeding techniques, and parental employment status [4,5]. Nonorganic FTT mostly occurs in infants under one year of age. In some cases, FTT is identified to be multifactorial or mixed, that is organic and nonorganic reasons occur in tandem. Most of the researches conducted in this connection reveal that factors such as not receiving sufficient nutrients, inappropriate diet, and infectious diseases, especially fever and diarrhea, which are common among children, are among the most common factors in growth failure development. As the findings of most studies indicate, diarrhea is more effective than other factors on the weight faltering of children [6-8]. Three kinds of FTT can be distinguished. In the first type, height and head diameter is normal but weight is below normal which can be due to undernutrition or an acute disorder. Weight is the most sensitive indicators in identifying FTT. If child's body does not receive enough nutrients, it uses fat or even muscle mass to provide the required nutrients. Decreased weight is often indicative of malnutrition. However, it should be noted that children lose weight within the first 10-15 days after their birth which can be attributed to losing body water; thus, it is a normal condition. In the second type of FTT, head diameter is normal but weight and height are below normal which might be due to endocrine disorders, genetically short stature or bone dystrophy. Finally, in the third type of FTT, weight, height, and head diameter are below normal. This type of FTT results from intrauterine growth retardation, nervous system defects, intrauterine infections, etc. If both parents have short stature, child is very likely to have genetically small stature and it cannot be attributed to malnutrition. However, in some studies, malnutrition is introduced as the leading cause of FTT which primarily affects children's weight and then, their head diameter [9]. Nutritionists can also be considered to be effective in that they can provide parents with constructive advice regarding children's nutrition and feeding techniques and their appropriate feeding time [10]. In the present study, in line with the bulk of research conducted in this regard, weight disorder is considered as the growth failure indicator. However, investigating other factors such as parent educational level, parental employment status, number of children, frequent attendance and care, and mother's general knowledge regarding childcare is also of paramount importance [1]. Growth failure is a global problem and, according to the World Health Organization statistics, more than 30% of children younger than two years of age are afflicted with growth failure, and, from among these children, 80% have decreased height growth and 20% are underweight [11]. According to the conducted researched in this field, FTT is more common among children in developing countries and, in most of these countries, children and infants' physical growth is below the international standards. To the researchers' best knowledge, the last piece of research carried out in this area in Iran dates back to 1998. The findings of this study revealed that 12.8% of children under five years of age are stunted for their age, 13.7% are severely or moderately underweight, and 4.8% are underweight. From birth, as age increases, the level of being underweight increases such that, at the age of two, it reaches its peak. Given the prevalence of growth failure and the adverse effect it might have on children's future, more research needs to be carried out in this connection [6]. Most of the data in the fields of economics, humanities, and biology are clustered or hierarchical. Hereditary studies, for example, deal with hierarchical data in that children are grouped into families. Children of the same family are similar to each other in terms of physical and mental characteristics, compared to members of other populations. One of the important factors in hierarchical data is the correlation between observations and, accordingly, the assumption of independence of observations is not satisfied [12-14]. Longitudinal studies which are extensively used in the field of medicine, economy, psychology, and behavioral science are developing and can be subsumed under the rubric of this type; thus, the independence of observations assumption is not satisfied for longitudinal and repeated measures data. One of the distinctive characteristics of longitudinal studies is repeated measurement of various subjects during different time points and the time effects and variations for each subject can be separated from group effects [15-18]. Given that there is a correlation between repeated measures, using common regression models whose major premise is independence of observations, leads to biased estimates with low precision. Accordingly, to analyze the data, models, taking the hierarchal structure and correlation between data into consideration, should be deployed [19-24]. Multilevel methods are generalizations of the generalized linear models in which, besides the response variable, regression coefficients are modeled. This method is an effective method in modeling nested and longitudinal data and aims at modeling the response variable as a function of independent variable in more than one level [25-29].

In the present study, the collected data were longitudinal and they were characterized by lack of independence of observations and correlation between a child's measures during different time points. This characteristics prohibits using common statistical methods such as the linear regression model. On the other hand, although factors affecting growth failure are identified, it cannot be predicted precisely in that children's resistance to disease, and genetic, family and environmental factors have proven to play a crucial role in having growth failure. With the above remarks in mind, the present study attempts at investigating growth failure in children under five years of age. Accordingly, to analyze the longitudinal data, multilevel regression model is used.

2 Data

The data of the present study were collected from children under five years of age who had referred to health centers in Isfahan. 1500 children were selected using random cluster sampling. In this connection, five centers were selected according to visit volume of each cluster of samples.

3 Multilevel regression model

In longitudinal studies, data structure is hierarchical with two levels such that different measures of participants constitute the first and second level. In this type of data, the assumption regarding independence of observations is not satisfied; accordingly, appropriate models, in which correlation between observations is taken into consideration, should be used. Multilevel regression model, which is also known as hierarchical linear model, is one of the most effective models for the analysis of longitudinal data and has recently received unprecedented attention [21-25]. In the multilevel regression model, it is assumed that participants are measured during different time points (t) for n_i times and $t = 1, \dots, n_i$. The response variable y_{it} indicates the measured value for the ith participant at t time point. This model can be considered as the following:

$$y_{it} = (\beta_{00} + \beta_{01}x_i + \beta_{10}z_{it} + \beta_{20}t_i + \beta_{11}x_iz_{it} + \beta_{21}x_it_i) + (u_{i0} + u_{i1}z_{it} + u_{i2}t_i + \epsilon_{it})$$

where $\beta_{00}, \beta_{01}, \beta_{10}, \beta_{20}, \beta_{11}$ and β_{21} are the regression coefficients and u_{i0}, u_{i1} and u_{i2} refer to the random effects model. As it can be observed, the above-mentioned mixed method involves two parts, namely the fixed effects and random effects model [24, 25]. To describe the data, the multilevel modeling was deployed. As for scrutinizing growth failure, the SAS software was used.

4 Results and Findings

In the present study, 57.6% of the population (864 participants) were female and 42.4% (636 children) were male. Mothers' mean age at birth was 26.4 ± 6.3. In addition, 59.2% of mothers (888 mothers) were housewives and 46.8% (702 mothers) of mothers had not high school diploma. According the findings of chi square test, there is a significant relationship between mothers' education level and failure to thrive (p=0.225). Mean ($\pm$ standard deviation) of children's birth weight was 3107.13±396.12 gram suggesting that 26.13% of them were underweight (had less than 2500 grams). The results also showed that almost 70% (1050 of them) of children had growth disorders during some time period until two years old. Growth failure condition of these children to two years of age is provided in Table 1. Frequency distribution of children for different diseases is depicted in Table 2 and Figure 1.

Table 1. Frequency distribution of the number of growth failure occurrences.

Number of growth failure occurrences	*Number*	*Percent*
Infant	450	30
One time	629	42
Two times	317	21
Three times	89	6
Four times or more	15	1

Table 2. Frequency distribution of children with growth failure categorized based on suffering from different diseases.

Suffering from disease	*having growth failure*	*not having growth failure*
	Number (Percent)	*Number (Percent)*
with infection	11(0.7)	6(0.4)
without ifection	1489(99.3)	1494(99.6)
with diarrhea	182(12.1)	29(1.9)
without diarrhea	1318(87.9)	1471(98.1)
suffering from other diseases	93(6.2)	17(1.1)
not suffering from other diseases	1407(93.8)	1483(98.9)

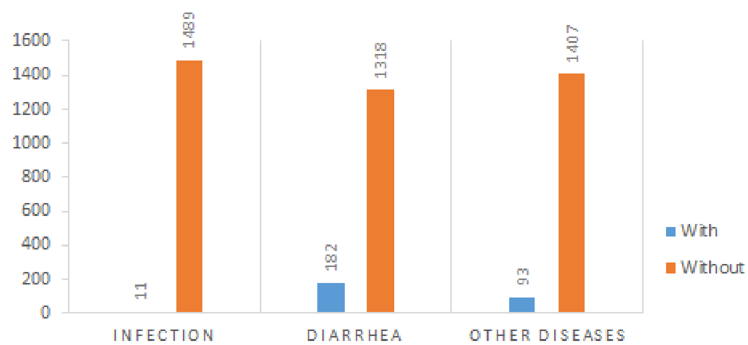

Figure 1: Frequency chart of children with growth failure categorized on suffering from fifferent diseases.

According to the multilevel regression model results, there is a significant relationship between growth failure and factors affecting growth failure such as individual differences and genetic factors (the random effects model), cold factors, nutrition, teething, infection, diarrhea, stopping breast feeding, and suffering from other related diseases (p¡0.001). However, fever does not have a significant relationship with failure to thrive. To investigate the unexamined and random effect of individual factors which are characteristics of each child and the effect of time on growth failure development, random components of model are scrutinized. Final results of multilevel regression model along with odds ratio estimates are presented in Table 3.

Table 3. Estimates of the parameters of multilevel model considered in investigating the effect of different factors on children's growth failure.

Variable	*Estimation*	*Standard Deviation*	*Odds Ratio*
diarrhea	0.801	0.110	2.21
teething	0.402	0.017	1.60
nutrition	0.509	0.016	1.75
stopping breast feeding	0.441	0.027	1.56
having cold	0.459	0.013	1.65
infection	0.352	0.042	1.46
other diseases	0.716	0.033	1.86
time random effect	0.011	0.004	-
intercept random effect	0.014	0.009	-
model's error	0.261	0.001	-

According to this table, variance components (the random effects model) of both the intercept and time are obtained to be significant. The significance of the random part of intercept indicates that the children under investigation in the present study had different primary value;

hence, individual differences and genetic factors can be conceived of as effective factors in developing growth disorders. To put it differently, two child who are identical in terms of nutritional conditions and suffering from diseases, behave differently regarding being underweight. The random component of time or the slope of the model, on the other hand, suggests that the probability of developing growth disorder is different during different time points. According to odds ratio estimates, compared to other factors, diarrhea, suffering from illnesses, and the nutrition factor are more effective and have the lion's share in developing growth disorders.

5 Discussion and Conclusion

The present study made an attempt to shed some light on the relationship between effective factors leading to growth failure using multilevel regression model. The results demonstrate that stopping breast feeding, teething, and urinary and respiratory tract infection, are the most effective factors in developing growth disorders. In most of previously-conducted studies, it is argued that infectious diseases are the initiating factor in impairing children's weight gain [30-32]. In this connection, Rawland and Timothy demonstrated that diarrhea, urinary and respiratory tract infection, and fever are the most effective factors in increasing the risk of being underweight. The authors further illustrated that, from among these factors, diarrhea is the most important factor [31]. However, in the same line of research, Smith and Kolseren demonstrated that urinary and respiratory tract infection exerts a more adverse effect on being underweight and developing growth disorders [33,34]. Furthermore, Khodali et. al concluded that stopping breast feeding and diarrhea are the most crucial factors leading to growth failure. In the present study, teething is proved to be an effective factor with big odds ratio, however, in Khodali et. al's study, this factor is concluded to play a less crucial role [4]. In classifying malnutrition reasons, UNICEF identifies and reports receiving inadequate nutrition and suffering from diseases as effective factors having rapid and decisive effect on children nutrition condition. Furthermore, this finding is corroborated by UNICEF reports declaring that, according to epidemiologic evidences, children's primary response to nutritional problems and infections is losing weight [35,36]. However, quite contrary to UNICEF reports, De Villiers showed that nutritional problems and diseases do not have direct effect on weight loss or growth failure disorders [37]. The results of Ehsanipour's study, likewise, confirmed that diarrhea is one the major and principal reasons leading to weight loss. Furthermore, it was demonstrated that as mothers become more educated and their educational level increases, children's weight loss decreases [38]. In the present study, no statistical relationship was observed between mothers' level of education and being afflicted with growth disorders; however, in some other studies conducted in the same line of research, a significant relationship is identified between mothers' level of education and growth disorders; such that as mothers' level of education increases, children's weight loss decreases [39]. In this connection, Bachner et. al contend that, in developing countries, organic factors such as suffering from acute or chronic diseases play a more crucial role in having growth disorders; while, in developed countries such as the united states of America, nonorganic factors such as environmental and mental factors are more important in growth failure development [40]. As it was corroborated in the present study, after diarrhea, stopping breast feeding is the most important factor affecting growth failure. After stopping breast feeding, children should be provided with complementary foods, thus, if complementary foods are inappropriate or are not provided at the right time, they can affect growth failure decisively. In other related studies, the significant relationship between stopping breast feeding and growth failure is confirmed and attested [22, 42-44]. In the present research, it was also revealed that urinary and respiratory tract infection have a significant effect of developing growth disorders. This finding is corroborated by the results of previously-conducted studies [20,43]. Besides the above-mentioned factors, we can take the effect of individual differences and genetic factors into account in that the multilevel regression model involves a random component. Since the results of the random effects model are significant, we can maintain that, besides all organic and nonorganic factors, individual characteristics of each child plays a determining role in developing growth failure. Accordingly, in one child, diarrhea might lead to weight loss and growth failure; while, in another child with different individual characteristics, it might not lead to losing weight. Taking this issue, which has not received the attention it deserves, into account and investigating it might have a far-reaching effect on the conclusions in this field. To put it in a nutshell, according to the findings of the present study, it can be concluded that suffering from diseases such as diarrhea, infection, and cold, along with nutritional factors such as stopping breast feeding are effective in developing growth disorders. In addition, increasing mothers' knowledge regarding childcare can be effective in reducing growth failure. Furthermore, unidentified factors and some of genetic and environmental factors are effective in developing growth disorders; accordingly, this leads to the conclusion that, in identical conditions, children develop different levels of growth disorder.

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Maintenance planning for a continuous monitoring system using deep reinforcement learning

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Abstract

This paper proposes a maintenance decision-making framework for multi-unit systems using Machine Learning (ML). Specifically, we propose to use Deep Reinforcement Learning (RL) for a dynamic maintenance model of a multi-unit parallel system that is subject to stochastic degradation and random failures. As each unit deteriorates independently in a three-state homogeneous Markov process, we consider each unit to be in one of three states: healthy, unhealthy, or a failed state. We model the interaction among system states based on the Birth/Birth-Death process. By combining individual component states, we define the overall system state. To minimize costs, we use the Markov Decision Process (MDP) framework to solve the optimal maintenance policy. We apply the Double Deep Q Networks (DDQN) algorithm to solve the problem, making the proposed RL solution more practical and effective in terms of time and cost savings than traditional MDP approaches. A numerical example is provided which demonstrates how the RL can be used to find the optimal maintenance policy for the system under study.

Keywords: Dynamic Maintenance, Manufacturing Systems, Deep Reinforcement Learning

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1 Introduction

The main difficulty in today's manufacturing systems, which are ever more complicated, is handling unforeseen breakdowns. The use of maintenance scheduling can increase productivity, improve product quality, and boost customer satisfaction by reducing costly repairs and replacements, minimizing downtime, and preventing unexpected breakdowns. Proper maintenance scheduling is a crucial component of modern manufacturing practices since it is a strategic method that maximizes resource use and minimizes production interruptions. There are several drawbacks to the traditional maintenance scheduling methods. Traditional maintenance planning may be performed either too regularly, which would result in excessive costs and downtime, or insufficiently frequently, which would increase the chance of unexpected breakdowns. On the other hand, modern maintenance methods such as condition-based monitoring can offer more accurate and cost-effective maintenance strategies [1].

Condition-based maintenance (CBM) is a state-of-the-art maintenance strategy that recommends maintenance action based on the information collected through condition monitoring. CBM program consists of three main steps: (i) Data acquisition, (ii) Data processing, and (iii) Decision making [6]. Two important aspects of a CBM program are diagnostics and prognostics. Intuitively speaking, there are two different maintenance methodologies for fault diagnosis and prognosis, namely (i) Process-Driven Models, and (ii) Data-Driven Models. Process-driven models rely on the statistical information which is extracted from the process. In other words, these processes can be explained by a series of mathematical (set of algebraic or differential equations) or physical equations. On the other hand, data-driven models use machine learning approaches to establish correlations between input and output data [4]. Researchers have been using statistical models for decades as a promising solution methodology for maintenance management.

Several research has investigated model-based maintenance in many sectors. Model-based maintenance has been used in the energy sector to monitor wind turbines and predict component breakdowns. It is used to forecast gearbox failures in wind turbines, which lowers maintenance costs and downtime, according to the research in [5]. There are various restrictions with model-based maintenance, one of which is that it requires accurate models of the system under observation. It may be time-consuming and costly to develop these models. Moreover, systems with unpredictable behavior or operating conditions that change quickly may not perform well with model-based maintenance. Since, statistical models may not be able to accurately predict future failures or maintenance needs, particularly in complex systems with many interacting components, the main focus of researchers is on creating cutting-edge Machine Learning (ML) and Artificial Intelligence (AI)-based solutions for maintenance management problems.

For instance, [2] studied a serial production line with intermediate buffers. If just one machine failed, the whole line will be stopped, so finding the optimal preventive maintenance policy is so crucial. A double deep Q-network algorithm is used to learn the PM policy. [8] proposed a deep RL-based CBM to overcome high-dimensional problems as well as low-dimensional ones. Both stochastic and economic dependencies are taken into consideration. They mapped the system degradation directly to the maintenance decision without the necessity of having a maintenance threshold. [3] proposed a novel RL approach that can handle various maintenance strategies without any prior knowledge while taking spare component storage costs into account. It's noteworthy that they created sufficient training data for an RL-driven strategy by simulating various maintenance scenarios. Using a deep reinforcement learning methodology, [7] introduced a new dynamic maintenance model for a deteriorating repairable system exposed to deterioration. Instead of discretizing the state space, they take into account the precise degree of system deterioration to represent their issue.

In this paper, we propose an RL-based model for CBM for a multi-unit parallel system to find the optimal policy for maintenance actions. Unlike the previous research in this context which focus on the degradation level of the system, the number of failed and unhealthy units is taken into consideration. The remaining sections of the paper are organized as follows: In Section 2, we first describe a general view of the problem and then we go through the details during this section. In section 3 we discuss the details of the numerical experiment.

2 problem formulation

2.1 System Description

Consider a production system consisting of M identical, parallel units. Deterioration of each unit occurs independently according to a three-state homogenous Markov process such that each unit has three states: healthy, unhealthy, and failure denoted as states 0, 1, and 2, respectively. Healthy and unhealthy states are operational states and the failure state is not operational. Each unit of the system monitors continuously so that updated information about the deterioration of each unit is available. At the start of the planning horizon, all units are in a healthy state and gradually deteriorate. The system state is defined using two interacting populations: (i) The number of units in the unhealthy state, and (ii) The number of units in the failure state. More specifically, at a specific time as t, the state of the system is defined as $s_t = (i, j)$. So that i is the number of units in the failure state, and j is the number of units in the unhealthy state and $0 \leq i + j \leq M,\ i, j \geq 0$. It means that the number of units in the healthy state is $M - i - j$. Accordingly, the total number of system states is equal to $(M + 2)(M + 1)/2$. The Birth/Birth-Death stochastic process is employed to describe the interaction among the system states, for more details see [1].

2.2 Markov Decision Process

Generally speaking, ML models can be classified into three major categories: (i) Supervised learning, (ii) Unsupervised learning, and (iii) Reinforcement learning. In an RL model, an agent interacts with its environment to discover the best action to take in a given state from a set of possible states. After getting an action, the agent will get feedback from the environment. The feedback can be good (reward) or bad (penalty). This whole process is known as the Markov Decision Process (MDP), which includes a set of actions A, a set of states S, a transition matrix P, and a reward function, denoted by R. Agent and environment are two main components of RL. At every time step t, the agent is in state $s_t \in S$ and will take an action $a_t \in A$. Then, based on transition probabilities it will go to state $s_{t+1} \in S$ with a reward of $r_t = R(s_t, a_t)$. This process continues until the agent reaches the terminal state. Ultimate goal of an MDP is determining the optimal policy in order to maximize (minimize) the expected returns (costs) while the stages can be finite or infinite. Hence, policy is a fundamental concept in MDP and consequently in RL and DRL. Given a specific state of MDP, a policy prescribes which action should be selected among the eligible actions of that state. A policy fully describes the behavior of the agent in MDP. Given that the system is in state s, under the policy π, the probability of taking action a is $P(A_t = a|S_t = s)$. Hence, a policy is a map from the states of the system to the action set: $\pi : S \rightarrow A$.

The optimal policy denoted as π^* is the one that results in the maximum accumulation of rewards through interactions. In other words, the main goal of RL agent is reward maximization. The output of every RL problem is the optimal policy, denoted by π^*, which leads to achieving the maximum accumulated rewards during interactions, expressed as:

$$\pi^* = \arg\max_{\pi} E_{\pi}\Big\{\sum_{t=0}^{H-1} \gamma^t r_{t+1}|s_0 = s\Big\},$$

where $\gamma \in [0, 1]$ and H represent the discount factor and the number of finite episodes in MDP. While the low value of γ maximizes the short-term rewards, a higher one leads to increasing the long-term rewards.

2.3 Deep Reinforcement Learning

Action-value function is the expected return given that the system is in state s, action a is taken, and hereafter policy π is employed. It is denoted by $Q_\pi(s, a)$. In DRL, action-value function can be parametrized in order to approximate/estimate the true value of action-value function. Given the parameter θ, the action-value function under policy π is approximated as follows:

$$Q(s, a, \theta) \approx Q_\pi(s, a).$$

Hence, $Q(s, a, \theta)$ is approximated action-value function parametrized with trainable parameter θ. There exists different function approximators such as linear combination of features, neural networks, and decision trees. Deep Q networks (DQN) is a prevalent method to train parameter θ and consequently finds a suitable approximation for true action value function. As its name indicates, it employs deep neural networks to find optimal values for θ. It is stated by some researchers that standard DQN algorithms suffer from overestimation for action-value function. Hence, to address the problems of standard DQNs, DDQN is proposed.

Parameters of online network and target network are denoted as θ and θ', respectively. Target network is used for policy evaluation, while online network is used for selecting actions given the current state. The parameter of online network is updated in each training step, but θ' of the target network is frozen and only after specified number of iterations, parameter of online network is copied into the target network. The goal of this point is stabilizing the learning process. It is used batch training to train the neural network. Previous transition steps are recorded in a reply memory. A minibatch of transitions from the reply memory is randomly selected to train the online network. Training of the online network means that an optimization algorithm is conducted to minimize the squared loss, i.e., $[Q_{target} - Q(s, a, \theta)]^2$, with respect to parameter θ . In each training step, according to the gradient descent method, the parameter is updated using the following equation:

$$\theta \leftarrow \theta + \alpha[Q(s, a; \theta) - Q_{target}]\nabla_\theta Q(s, a; \theta),$$

where α is learning rate.

2.4 Deep Reinforcement Learning for Maintenance Policy

We provide a DDQN approach for the CBM policy optimization of the system described in the previous section. Let define each component of RL in detail:

- Agent: The management system, as the intelligent component of the system, acts as the agent and interacts with the environment based on a set of maintenance decisions.

- State-space: In the proposed problem, a component could be in three different states: (i) Healthy state, (ii) Unhealthy state, and (iii) Failed state. Therefore, the state of the system is in the form of (N_2, N_1) which N_2 represents the number of failed units and N_1 represents the number of unhealthy units.

- Action-space: At each decision epoch, four actions are available. a_0: do nothing, a_1: conducting reactive maintenance (RM) on the failed units, a_2: conducting preventive maintenance (PM) on the unhealthy units, and a_3: conducting PM on the unhealthy units and RM on the failure units.

- Reward: Consider the following cost components in the model:

  - C_0, C_1: Operating cost rates of each unit in states 0 and 1, respectively.
  - C_F: Failure replacement cost of each unit.
  - C_P: Preventive maintenance cost of each unit.
  - C_D: Cost rate of lost production for unsatisfied demands. There is lost production if the total production rate of the system $PR = N_0 \times p_0 + N_1 \times p_1$ drops below the demand rate (D) at time t.
  - C_E: Profit rate obtained from excess production.
  - C_K: Fixed set-up cost including the cost of sending maintenance crew to perform maintenance.

  The reward function in the proposed model is a function of the cost components given by:

$$\text{Reward} = R_1 + R_2,$$

  where

$$\begin{aligned} R_1 &= (C_K + C_F \times N_2) \times B_1 - (C_K + C_P \times N_1) \times B_2 \\ &\quad -(C_K + C_F \times N_2 + C_P \times N_1) \times B_3, \end{aligned}$$

  in which B_1, B_2, and B_3 are indicator variables, and

$$\begin{aligned} R_2 &= -(M - N_1^{'} - N_2^{'}) \times C_0 - N_1^{'} \times C_1 \\ &\quad -C_D \times \max(0, D - ((M - N_1^{'} - N_2^{'}) \times p_0 + N_1^{'} \times p_1)) \\ &\quad +C_E \times \max(0, ((M - N_1^{'} - N_2^{'}) \times p_0 + N_1^{'} \times p_1) - D), \end{aligned}$$

  where $N_2^{'}$ is equal to zero if a_1 is selected, $N_1^{'}$ is equal to zero if a_2 is selected, and $N_1^{'}$ and $N_2^{'}$ are equal to zero if a_3 is selected.

3 Numerical Study

Consider a system with M=30 units. The agent will start from state $(0,0)$ and choose an action. Every single action changes the state of the system. The agent moves among states based on the Birth/Birth-Death process. The input parameters in Table 1 are considered in our model. Assuming $\gamma = 0.5$, $\alpha = 0.01$, the initial value for $\varepsilon = 0.9$ and using 64 neurons in the hidden layers of the networks in 2000 thousand episodes, the results of the DDQN implementation are shown in Figure 1, which indicates the convergence of the algorithm.

Table 2 shows some examples of agent performance in different states. For example, in States $(1,2)$ and $(2,3)$ when the number of healthy units is more, the agent has decided to choose action a_0, and in States $(6,9)$ and $(9,3)$, when the number of healthy units has decreased, the agent has decided to choose action a_3 which seems that it is cost-effective, in addition to the corrective repair of failed units, preventively replace unhealthy units because of high setup cost.

Table 1. The Value of Cost Components.

Parameter	D	p_0	p_1	C_0	C_1	C_D	C_E	C_K	C_F	C_P	M
Value	9000	450	230	5	10	0.8	0.04	500	5	1	30

Table 2. The Value of Cost Components.

States	$(1,1)$	$(2,1)$	$(1,5)$	$(3,7)$	$(2,5)$	$(6,9)$	$(7,8)$	$(9,3)$	$(9,4)$	$(8,5)$	$(7,7)$
Action	a_0	a_0	a_0	a_0	a_0	a_3	a_3	a_3	a_3	a_3	a_3

4 Conclusion

In conclusion, we proposed a deep RL-based framework for maintenance decision making with the goal of cost minimization. We considered a large parallel multi-unit system. Units are subject to random failures and are independent from each other. Each unit could be in three different states, namely, healthy, unhealthy and failed. Since, the number of units is very large which leads to a large state-space, we used a DDQN algorithm to obtain the optimal maintenance policy for the system. At the end, we provided a numerical example to evaluate the effectiveness of the proposed model compared to the traditional methods.

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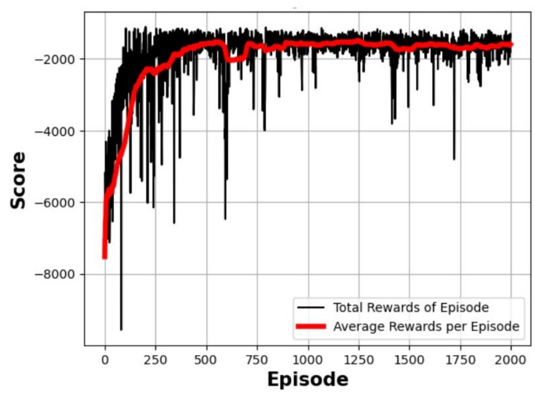

Figure 1: Reward Function

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Confidence regions for the parameters of an inverted exponentiated Pareto distribution based on progressively Type-II censored samples

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Abstract

In this article, based on progressively type-II censored order statistics, we create balanced confidence regions (BCR) and optimal confidence regions (OCR) for the parameters of an inverted exponentiated Pareto (IEP) distribution. Constraint optimization problem and nonlinear programmings methods are used in order to construct OCRs. Monte Carlo simulation studies are used to evaluate the performance of the methods proposed in this paper. Finally, a numerical example is presented to illustrate the proposed regions.

Keywords: Confidence region, Inverted exponentiated Pareto distribution, Progressively Type-II censored, Optimization, Monte Carlo.

1 Introduction

The two-parameter exponentiated Pareto distribution is a lifetime distribution and is widely used in many fields. This distribution is widely used in reliability and life-testing studies. In many life-testing and reliability studies, lifetime experiments are usually terminated prior to

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the failure of all experimental units in the sample. Under conventional censoring schemes, the experimenter can not remove (or censored) units from a life-testing at various stages during the experiments. Progressive Type-II censoring scheme is a more general censoring scheme in which surviving units can be removed from the study at any time of failures (see, e.g., Balakrishnan and Aggarwala [2], Balakrishnan and Cramer [3]).

Suppose that lifetime X of items follow an IEP distribution with cumulative density function (CDF) given as

$$F(x;\alpha,\lambda) = 1 - \left[1 - \left(\frac{x}{1+x}\right)^{\lambda}\right]^{\alpha}, \qquad x, \lambda, \alpha > 0, \tag{1}$$

and the probability density function (PDF) is given by

$$f(x;\alpha,\lambda) = \alpha\lambda \frac{x^{\lambda-1}}{(1+x)^{\lambda+1}} \left[1 - \left(\frac{x}{1+x}\right)^{\lambda}\right]^{\alpha-1}, \qquad x, \alpha, \lambda > 0, \tag{2}$$

where λ and α are both shape parameters.

In the issue of confidence intervals, Dey and Dey [4] provided the approximate confidence interval of the generalized inverted exponential (GIE) distribution based on a progressively censored sample. We know that when the sample size is small, the importance of calculating the exact confidence intervals is more understandable. Recently, Kinaci et al. [6] introduced the exact confidence intervals and regions for GIE parameters and studied based on progressively type-II censored sample and recorded values. Also, Wang et al. [8] obtained the confidence set for the GIE parameters based on k-record values.

Confidence regions can be used to find confidence bands for a bivariate function of the scale and shape parameters in IEP distribution, such as the reliability and cumulative distribution functions. Further, these regions are useful for testing hypotheses related to the parameters of a general class of inverse exponentiated distributions.

The rest of the paper is organized as follows: In Section 2 and 3, based on the progressive type-II censoring, we obtain balanced and optimal confidence regions for the parameters of IEP distribution. In Section 4, a simulation study is performed to compare the areas of optimal and balanced confidence regions and it is shown that the reduction in areas of the optimal joint confidence regions with respect to the balanced confidence regions is substantial. a numerical example is given in Section 5 for illustrative and comparative purposes. Finally, some concluding remarks are presented in Section 6.

2 The balanced confidence region

Let $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:n:n}$ denote a progressively Type-II censored sample from IEP distribution with a pdf in Equation (2) with censoring scheme $(R_1, \ldots, R_m)$.

Let

$$Y_{i:m:n} = -\alpha \log \left[1 - \left(\frac{X_{i:m:n}}{1+X_{i:m:n}}\right)^{\lambda}\right], \; i = 1, 2, \ldots, m.$$

For notation simplicity, we will write X_i for $X_{i:m:n}$. It can be seen that $Y_{1:m:n} < Y_{2:m:n} < \cdots < Y_{m:m:n}$ are progressively type-II right censored order statistics from a standard exponential distribution. For notation simplicity, we will write Y_i for $Y_{i:m:n}$. Let

$$\eta_i = n_i(Y_i - Y_{i-1}), i = 2, \ldots, m$$

where $n_i = n - \sum_{s=1}^{i-1}(1+R_s), i = 2, \ldots, m$ and $\eta_1 = nY_1$. It is well known that, from Thomas and Wilson [9], $\eta_1, \eta_2, \ldots, \eta_m$ are independent and identically distributed as a standard exponential distribution.

For $j = 1, 2, \ldots, m-1$, define

$$U_{(j)}(\lambda) = \frac{\sum_{i=1}^{j}\eta_i}{\sum_{i=1}^{m}\eta_i} = \frac{\sum_{i=1}^{j}(1+R_i)\log\left(1-\left(\frac{X_i}{1+X_i}\right)^{\lambda}\right) + n_{j+1}\log\left(1-\left(\frac{X_i}{1+X_j}\right)^{\lambda}\right)}{\sum_{i=1}^{m}(1+R_i)\log\left(1-\left(\frac{X_i}{1+X_i}\right)^{\lambda}\right)}. \quad (3)$$

It can easily be shown that function $U_{(j)}$ is strictly descending from λ. Moreover, $\lim_{\lambda\to 0} U_{(j)}(\lambda) = \infty$ and $\lim_{\lambda\to\infty} U_{(j)}(\lambda) = 0$. Thus, if $u > 0, U_{(j)}(\lambda) = u$ has a unique solution for any $\lambda > 0$. Hence $U_{(j)}^{-1}(.)$ are strictly decreasing. It is observed that $U_{(1)}, U_{(2)}, \ldots, U_{(m-1)}$ are order statistics from the uniform (0,1) distribution with sample size $m-1$.

To obtain a balanced joint confidence region under a progressive Type II censoring scheme for the parameters α and λ, define

$$T(\lambda) = -2\sum_{j=1}^{m-1}\log\left(U_{(j)}\right),$$

Furthermore,

$$T(\lambda) = -2\sum_{j=1}^{m-1}\log\left(\frac{\sum_{i=1}^{j}(1+R_i)\log\left(1-\left(\frac{X_i}{1+X_i}\right)^{\lambda}\right) + n_{j+1}\log\left(1-\left(\frac{X_i}{1+X_j}\right)^{\lambda}\right)}{\sum_{i=1}^{m}(1+R_i)\log\left(1-\left(\frac{X_i}{1+X_i}\right)^{\lambda}\right)}\right), \quad (4)$$

According to Viveros and Balakrishnan (1994) [7], T has a chi-square distribution with $2(m-1)$ degrees of freedom. Based on above results, $T(\lambda)$ increases in λ.

The second pivotal quantity Z can be defined as

$$Z(\alpha, \lambda) = 2\sum_{i=1}^{m}\eta_i = \alpha W(\lambda),$$

where

$$W(\lambda) = -2\sum_{i=1}^{m}(R_i+1)\log\left(1-\left(\frac{X_i}{1+X_i}\right)^{\lambda}\right). \quad (5)$$

Z has a chi-square distribution with $2m$ degrees of freedom. Moreover, $U_{(j)}$ and Z are stochastically independent. Also, T and Z arestochastically independent.

In order to obtain the BCR for the parameters of IEP distribution, a useful theorem is given first as follows.

Theorem 2.1. *Suppose that $X_1 < X_2 < \cdots < X_m$ is a progressively Type-II censored sample from IEP distributions with a pdf in Equation (2) with censoring scheme $(R_1, \ldots, R_m)$. Based on the pivotal quantities T and Z, a $100(1-\gamma)\%$ joint confidence region for (α, λ) is given by*

$$C_{1-\gamma}^{BCR} = \left\{(\alpha,\lambda) : \frac{\chi^2_{2m,p_1}}{W(\lambda)} < \alpha < \frac{\chi^2_{2m,p_2}}{W(\lambda)}, T^{-1}(\chi^2_{2(m-2),p_1}) < \lambda < T^{-1}(\chi^2_{2(m-2),p_2})\right\}, \tag{6}$$

where $p_1 = (1-\sqrt{1-\gamma})/2$ and $p_2 = (1+\sqrt{1-\gamma})/2$ and for $0 < p < 1, \chi^2_{\nu,p}$ denotes the p^{th} quantile of χ^2_ν. Also, $T^{-1}(t)$ is the solution of λ for the equation $T(\lambda) = t$.

The area of $\mathrm{C}_{1-\gamma}^{BCR}$ is given by

$$|\mathrm{C}_{1-\gamma}^{BCR}| = (\chi^2_{2m,p_2} - \chi^2_{2m,p_1}) \int_{T^{-1}(\chi^2_{2(m-2),p_1})}^{T^{-1}(\chi^2_{2(m-2),p_2})} \frac{\mathrm{d}\lambda}{W(\lambda)}. \tag{7}$$

3 The optimization problem

In this section, an optimal confidence region for (α, λ) based on the pivotal quantities (T, Z) is constructed. We used constrained optimization problems to construct the optimal confidence regions. The following theorem presents the optimal confidence region for (α, λ) based on the pivotal quantities (T, Z).

Theorem 3.1. *Let $X_1 < X_2 < \cdots < X_m$ denote a progressively Type-II censored sample from IEP distribution with censoring scheme $(R_1, \ldots, R_m)$. Based on the pivotal quantities T and Z, an optimal $100(1-\gamma)\%$ joint confidence region for (α, λ) is given by*

$$C_{1-\gamma}^{OCR} = \left\{(\alpha,\lambda) : \frac{b_1^*}{W(\lambda)} < \alpha < \frac{b_2^*}{W(\lambda)}, T^{-1}(a_1^*) < \lambda < T^{-1}(a_2^*)\right\}, \tag{8}$$

where a_1^*, a_2^*, b_1^* and b_2^* are computed as the solution of the optimization problem

$$\begin{aligned} &\textit{Minimize} \quad \int_{T^{-1}(a_1)}^{T^{-1}(a_2)} \frac{b_2 - b_1}{W(\lambda)} \mathrm{d}\lambda \\ &\textit{Subject to} \quad \Big(\Psi_{2(m-1)}(a_2) - \Psi_{2(m-1)}(a_1)\Big)\Big(\Psi_{2m}(b_2) - \Psi_{2m}(b_1)\Big) = 1-\gamma, \\ &\qquad\qquad\quad 0 < a_1 < a_2, \quad 0 < b_1 < b_2, \end{aligned}$$

where Ψ_ν is the cdf of $\chi^2(\nu)$ distribution with ν degree of freedom and $T^{-1}(t)$ is the solution of λ for the equation $T(\lambda) = t$.

Proof. Assuming that

$$P\left(a_1 < T(\lambda) < a_2,\ b_1 < Z(\alpha,\lambda) < b_2\right) = 1-\gamma,$$

or, equivalently,

$$\Big(\Psi_{2(m-1)}(a_2) - \Psi_{2(m-1)}(a_1)\Big)\Big(\Psi_{2m}(b_2) - \Psi_{2m}(b_1)\Big) = 1-\gamma,$$

the $100(1-\gamma)\%$ joint confidence region for (α, λ) is

$$\begin{aligned} \mathrm{C}_{1-\gamma} &= \left\{(\alpha,\lambda) : Z^{-1}\left(b_1,\lambda\right) < \alpha < Z^{-1}\left(b_2,\lambda\right), T^{-1}\left(a_1\right) < \lambda < T^{-1}\left(a_2\right)\right\} \\ &= \left\{(\alpha,\lambda) : \frac{b_1}{W(\lambda)} < \alpha < \frac{b_2}{W(\lambda)}, T^{-1}(a_1) < \lambda < T^{-1}(a_2)\right\}, \end{aligned}$$

where $T^{-1}(t)$ is the solution of λ for the equation $T(\lambda) = t$ and $Z^{-1}(z, \lambda)$ is the solution of α for the equation $Z(\alpha, \lambda) = z$. The area of $\mathrm{C}_{1-\gamma}$ is given by

$$|\mathrm{C}_{1-\gamma}| = \int_{T^{-1}(a_1)}^{T^{-1}(a_2)} \int_{Z^{-1}(b_1,\lambda)}^{Z^{-1}(b_2,\lambda)} \mathrm{d}\alpha \mathrm{d}\lambda = (b_2 - b_1) \int_{T^{-1}(a_1)}^{T^{-1}(a_2)} \frac{\mathrm{d}\lambda}{W(\lambda)}. \tag{9}$$

Thus, the optimal $100(1-\gamma)\%$ confidence region for (α, λ) based on the pivotal quantities (T, Z) is obtained by minimizing the area $|\mathrm{C}_{1-\gamma}|$ with respect to a_1, a_2, b_1 and b_2 subject to the constraint

$$\Big(\Psi_{2(m-1)}(a_2) - \Psi_{2(m-1)}(a_1)\Big)\Big(\Psi_{2m}(b_2) - \Psi_{2m}(b_1)\Big) = 1-\gamma,$$

for $0 < a_1 < a_2,\ 0 < b_1 < b_2$. This completes the proof. □

Then the area of $\mathrm{C}_{1-\gamma}^{OBC}$ is given by

$$|\mathrm{C}_{1-\gamma}^{OBC}| = (b_2^* - b_1^*) \int_{T^{-1}(a_1^*)}^{T^{-1}(a_2^*)} \frac{\mathrm{d}\lambda}{W(\lambda)}. \tag{10}$$

4 Simulation study

In this section, a Monte Carlo simulation is carried out in order to investigate the performance of the proposed BCR and OCR. In this simulation, we computed the BCRs and OCRs for (α, λ) as discussed in Sections 2 and 3. We selected the default values of the parameters and the size of the observations so that we can measure the behavior of the average area corresponding to the various confidence regions proposed with respect to increasing the parameter values or increasing the censoring levels $100(1 - m/n)\%$. We have designed four schemes for simulation with n and m as: Scheme I: $R = (\underbrace{0,\ldots,0}_{m-1}, n-m)$, Scheme II: $R = (n-m, \underbrace{0,\ldots,0}_{m-1})$, Scheme III: $R = (\frac{n-m}{2}, \underbrace{0,\ldots,0}_{m-2}, \frac{n-m}{2})$, Scheme IV: $R = (\underbrace{0,\ldots,0}_{m/2-1}, n-m, \underbrace{0,\ldots,0}_{m/2})$.

The reduction in size of the A set with respect to the corresponding B region is defined as $100(1 - |A|/|B|)\%$. The results for these simulations are summarized in the Tables 1 and 2.

Table 1. The average confidence area (ACA), coverage percentage (CP) and average reduction (AR) of the $C_{0.95}^{BCR}$ and $C_{0.95}^{OCR}$ confidence regions of (α, λ).

					ACA			*CP*	
λ	α	n	m	Scheme	BCR	OCR	AR	BCR	OCR
2	3	20	16	I	17.8557	14.5376	%14	0.951	0.952
				II	12.3410	10.9135	%10	0.949	0.948
				III	16.1773	13.7778	%13	0.948	0.947
				IV	13.4880	11.7344	%11	0.950	0.949
			18	I	11.0275	9.6934	%11	0.951	0.948
				II	9.2741	8.3548	%9	0.947	0.952
				III	12.2495	10.8086	%10	0.948	0.947
				IV	11.3048	10.0242	%10	0.947	0.951
		24	18	I	16.2205	13.5204	%13	0.948	0.947
				II	11.8898	10.6242	%9	0.951	0.950
				III	11.9122	10.3986	%11	0.947	0.947
				IV	9.4506	8.4904	%10	0.951	0.952
			20	I	10.6589	9.3584	%11	0.948	0.951
				II	9.9473	8.9581	%8	0.952	0.950
				III	9.6732	8.6608	%10	0.948	0.947
				IV	9.2351	8.3264	%9	0.951	0.952
	5	20	16	I	30.9496	24.4323	%17	0.947	0.951
				II	29.1180	24.2763	%13	0.950	0.952
				III	25.7408	21.2743	%15	0.949	0.951
				IV	23.0614	19.4474	%13	0.948	0.947
			18	I	24.8779	20.3632	%13	0.949	0.947
				II	22.8897	19.3833	%12	0.950	0.952
				III	19.0571	16.3070	%12	0.948	0.948
				IV	23.9572	20.1493	%12	0.949	0.946
		24	18	I	28.0465	22.1242	%16	0.951	0.949
				II	19.3313	16.8727	%11	0.952	0.951
				III	33.2771	26.1985	%14	0.948	0.952
				IV	19.6688	16.9584	%12	0.949	0.951
			20	I	18.3311	15.5316	%13	0.950	0.951
				II	18.1261	15.9342	%11	0.952	0.947
				III	18.2151	15.7105	%12	0.948	0.950
				IV	12.7028	11.3168	%10	0.949	0.948

Table 2. The average confidence area (ACA), coverage percentage (CP) and average reduction (AR) of the $C_{0.95}^{BCR}$ and $C_{0.95}^{OCR}$ confidence regions of (α, λ).

λ	α	n	m	Scheme	*ACA*		AR	*CP*	
					BCR	OCR		BCR	OCR
4	3	20	16	I	43.1320	34.5983	%14	0.951	0.949
				II	25.7450	22.8862	%10	0.947	0.948
				III	30.4935	25.9927	%12	0.951	0.947
				IV	27.8593	24.3287	% 11	0.952	0.951
			18	I	30.1782	25.6210	%11	0.949	0.952
				II	31.0517	26.4580	%10	0.951	0.946
				III	24.5839	21.5966	%10	0.945	0.946
				IV	19.1328	17.1632	%9	0.951	0.948
		24	18	I	33.8629	28.0568	%13	0.949	0.946
				II	24.1492	21.6610	%9	0.948	0.947
				III	22.3439	19.4794	%11	0.950	0.953
				IV	19.4357	17.3446	%10	0.948	0.954
			20	I	20.4957	18.0679	%11	0.952	0.951
				II	16.6702	15.2384	%8	0.951	0.953
				III	20.8256	18.5031	%10	0.952	0.951
				IV	19.6430	17.6773	%9	0.947	0.950
4	5	20	16	I	50.6404	40.8817	%16	0.952	0.951
				II	45.5263	38.9142	%13	0.948	0.947
				III	52.5354	43.3546	%15	0.951	0.948
				IV	43.0075	36.4701	%13	0.947	0.947
			18	I	47.6630	38.8726	%13	0.945	0.950
				II	37.4110	32.3423	%11	0.950	0.951
				III	46.7562	39.6677	%13	0.948	0.947
				IV	31.5138	27.6902	%11	0.948	0.949
		24	18	I	46.4914	37.3444	%15	0.951	0.950
				II	43.0423	37.2684	%11	0.947	0.952
				III	52.4100	43.0182	%14	0.948	0.947
				IV	31.5166	27.5685	%11	0.947	0.949
			20	I	58.5238	45.9006	%13	0.948	0.947
				II	36.7140	32.1504	%10	0.947	0.953
				III	40.0328	34.1497	%12	0.949	0.951
				IV	33.5528	29.3308	%10	0.948	0.946

Based on the progressive type-II censoring, the simulation results in Tables 1 and 2, show that the coverage probabilities of the joint confidence regions for (λ, α) are close to the desired level of $1 - \gamma$. Also, we observe that with increasing number of observations, the average area of related confidence regions decreases. The average area of the confidence regions can be considered as an increased function relative to the censoring levels. Of course, increasing the parameter values also leads to an average area increase in the corresponding confidence regions. We also observe that the area reduction of the optimal confidence regions for the corresponding balanced regions increases as m decreases or the level of censorship increases. A comparison of the average area of the confidence regions indicates that in all cases considered, when the degree of censoring increases, the optimal confidence region increases significantly compared to the balanced confidence region.

5 Application to real data

In this section, we discuss the confidence region proposed in this paper using a real data example.

Example 5.1. In this section, with the help of a set of real data from Zhu (2010)[10] in the context of the lifetime of 21 lamps from a constant-stress test , we will obtain the balance and optimal areas for the IEP parameters. These data (By dividing its limitation observation 130.47) are: 0.0267, 0.0371, 0.0661, 0.0683, 0.0715, 0.1469, 0.1505, 0.1564, 0.2084, 0.2164, 0.3115, 0.3216, 0.3770, 0.3948, 0.4273, 0.5487, 0.5752, 0.7065, 0.7843, 0.7898 and 0.9225. Based on the above observations, the MLEs of α and λ are estimated as $\hat{\alpha} = 6.7873$ and $\hat{\lambda} = 1.4747$, and the Kolmogorov-Smirnov distance is 0.11933 with an associated p-value of 0.8918, which indicates that the IEP distribution can fit these data match correctly. Based on the initial lifetime of the bulb data, a group of progressively censored Type II samples with $n = 21$, $m = 16$, $R_1 = 5$, and $R_i = 0$, $i = 2, ..., 16$ is produced as follows. 0.0267, 0.0371, 0.0661, 0.0715, 0.1469, 0.1564, 0.2164, 0.3115, 0.3216, 0.3770, 0.3948, 0.4273, 0.5487, 0.7065, 0.7843 and 0.9225.

By Theorem 2.1 and Theorem 3.1, the 95% joint confidence region for α and λ is determined by the following inequalities:

$$C_{0.95}^{BCR} = \begin{cases} \frac{16.8213}{W(\lambda)} < \alpha < \frac{52.4847}{W(\lambda)} \\ 0.7327 < \lambda < 2.2066 \end{cases} \text{ and } C_{0.95}^{OCR} = \begin{cases} \frac{15.2321}{W(\lambda)} < \alpha < \frac{52.1584}{W(\lambda)} \\ 0.5540 < \lambda < 2.0587 \end{cases}.$$

The area of $C_{0.95}^{BCR}$ is 12.3805, while the area of $C_{0.95}^{OCR}$ is 10.6843. The reduction in the size of $C_{0.95}^{OCR}$ concerning the corresponding $C_{0.95}^{BCR}$ is 13.70%, and the confidence regions $C_{0.95}^{OCR}$ and $C_{0.95}^{BCR}$ are plotted in Figure 1. As can be seen, the area of $C_{0.95}^{OCR}$ is significantly reduced compared to $C_{0.95}^{BCR}$ confidence

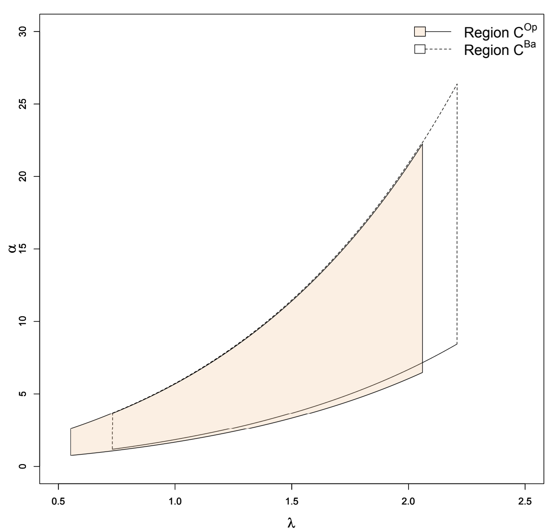

Figure 1: BCR and OCR (pink),$C_{0.95}^{BCR}$ and $C_{0.95}^{OCR}$, in Example 5.1.

region.

6 Concluding remarks

In this paper, we presented a classe of balanced and optimal confidence regions for parameters λ and α of IEP distribution based on progressive type-II censoring scheme. A constrained optimization problem was proposed in order to determine the optimal confidence regions for (λ, α). It is observed that the reduction in area of the optimal joint confidence region with respect to the balanced confidence regions is substantial.

The OCR presented in this paper can be used in hypothesis testing and estimation related to unknown parameters. For example, the p-value associated to the test of the null hypothesis $H_0 : (\alpha, \lambda) = (\alpha_0, \lambda_0)$ versus the alternative hypothesis $H_1 : (\alpha, \lambda) \neq (\alpha_0, \lambda_0)$ based on the smallest $100(1-\gamma)\%$ confidence region C^{OCR} for (α, λ) would be defined by

$$p = \max\{\gamma \in (0,1) : (\alpha_0, \lambda_0) \in C^{OCR}\},$$

see, e.g., Fernandez [5] and Asgharzadeh et al. [1].

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On the reliability of phased mission systems with non-identical components subject to external shocks

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Abstract

A phased mission system (PMS) accomplishes its mission if it completes different tasks successively. In fact, the PMS includes multiple consecutive, non-overlapping phases. In this paper, some representations based on the concept of survival signature are given for the reliability function of a PMS with non-identical components in three scenarios: (a) the component failures occur only based on their aging, (b) the component failures are only subject to external shocks and (c) the components failures are subject to both of aging, and external shocks. Finally, a numerical example is presented to explain the theoretical results.

Keywords: External shock, Phased mission system, Reliability, Survival signature.

1 Introduction

Phased mission systems (PMSs) have to perform a series of tasks in sequence to complete the determined mission. The operational life of such systems consists of a sequence of non-overlapping periods, called phases. The failure of the PMS in any phase causes mission failure. The PMSs appear in different areas of real-world applications, such as aerospace, communication networks, etc. An For example, the flight of an aircraft is a PMS with the phases of takeoff, cruise, and landing. Another example is

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a manufacturing system that performs a sequence of tasks by transferring a completed batch of parts from machine to machine [7]. The components of PMS may not have role in all phases and may be under the different pressure in different phases [15]. Therefore, we must consider both properties of changing the system structure and dependency between the phases to evaluate the PMSs reliability function. This made more complicate the study of PMSs reliability relative to other systems.

Numerous efforts have been made to evaluate the reliability of PMS since 1970s. Generally, all approaches can be classified into state space-oriented approaches and combinatorial models. There are also some models which combine two methods to have the advantages of both; see, e.g., [8], [9]. The state space-oriented methods usually use the continuous-time Markov chain, semi-Markov process, or Petri nets for estimation of the reliability of PMSs [13, 2, 3]. The state space-oriented methods provide flexible and powerful options for modeling complex dependencies among system components. However, the cardinality of the state space can become exponentially large as the number of components increases. The combinatorial methods use Boolean algebra, Binary Decision Diagram (BDD), multivalued decision diagram (MDD), or other decision diagrams to reduce the complexity of state space-oriented methods; see, e.g. [17, 6, 12]. Huand et. al. provided the exact closed form for reliability function of PMSs with binary components based on the new survival signature, [4]. The mechanism for computing survival signature is free of components' lifetimes. The reliability of PMSs with multi-type and multi-state components was evaluated under aging by using a new version of the survival signature, [1].

There are many situations in which the components may fail due to both factors of aging and external shocks. For example, highway bridges may collapse due to the failure of their components (aging) or may be subject to external shocks (such as earthquakes). So, we get incorrect results when the reliability is evaluated without considering the effects of external shocks. [5] investigated the reliability of PMSs under the assumption that the shocks cause a random amount of damage to the components. The reliability of coherent systems subject to aging and external shocks where each occurring shock destroys, with equal probabilities, one of the components functioning at the time of its occurrence evaluated by [11]. There are a few studies on the reliability of a PMS subject to aging and external shocks. The reliability of PMS subject to internal and external failures based on the BDD method is evaluated by [16]. See [10] and [14] for other research on the reliability of a PMS subject to aging and external shocks. In this paper, we evaluate the reliability of the PMS with non-identical components subject to aging and external shocks based on the concept of survival signature, and some compact representations are presented for the reliability function of a PMS. The proofs of theorems are omitted because of restrictions in page numbers.

2 Main results

For getting the results, the concept of meta-type components is used [4].

Definition 2.1. Components are defined to be of the same meta-type when they are of the same physical type and appear in the same phases.

Suppose that our PMS includes M phases with $\mathbb{K}$ different meta-type of n independent components where these components can be in two states: up state and down state. If we have m_k components of meta-type k, then $n = \sum_{k=1}^{\mathbb{K}} m_k$, $k = 1, 2, \ldots, \mathbb{K}$. Let phase i begin to operate at time τ_i and it ends at time τ_{i+1} where $i = 1, \ldots, M$, $\tau_1 = 0$ and τ_{M+1} is the full mission time. Note that at least one meta-type of components should be present in each phase. Suppose that the random failure times of components are statistically independent in the same phases and even that the components of the same meta-type can be exchangeable. Also, suppose that the components are non-repairable for the full duration time of the mission. Assume that X_{ij} denotes the state of component j, $j=1,2,...,n$, in phase i, $i=1,2,...,M$ where

$$X_{ij}=\begin{cases}1 & \text{if component } j \text{ functions for all of phase } i\\ 0 & \text{if component } j \text{ fails before the end of phase } i,\end{cases}$$

and $\mathbf{X}_i=(X_{i1},...,X_{in})$ shows the state vector of components in phase i and $\mathbf{X}^{(m)}=\big(\mathbf{X}_1,\ldots,\mathbf{X}_m\big)$, $m=1,...,\mathbb{M}$, gives the state vector of components for the first m phases. The state of PMS is also assumed to be binary and $\varphi^{(m)}$ denotes the state of PMS for the first m phases. Then, $\varphi^{(m)}(\mathbf{X}^{(m)})=1(0)$ means that the PMS works (fails) for the first m phases and the state vector $\mathbf{X}^{(m)}$. Let $\Phi_m(\ell_1,\ell_2,...,\ell_m)$ be the probability that the PMS works for the first m phases provided that ℓ_i components function in phase $i\in\{1,2,...,m\}$.

Under these assumptions, the survival signature of PMS for the first m phases is defined as follows [4]

$$\Phi_m(\ell_{1,1},...,\ell_{m,\mathbb{K}})=\left[\prod_{i=1}^{m}\prod_{k=1}^{\mathbb{K}}\binom{n_{i,k}}{\ell_{i,k}}\right]^{-1}\sum_{\mathbf{X}^{(m)}\in S^{(m)}}\varphi^{(m)}(\mathbf{X}^{(m)}),\tag{1}$$

where $\Phi_m(\ell_{1,1},...,\ell_{m,\mathbb{K}})$ denotes the probability that the PMS functions for the first m phases, $m=1,2,...,M$, given that $\ell_{i,k}$ components of meta-type k work in phase i, $i=1,2,...,m$, $k=1,2,...,\mathbb{K}$. Also, $S^{(m)}$ denotes the set of all possible state vectors of components state for the first m phases given that $\ell_{i,k}$ components of meta-type k function in phase i. Note that $n_{i,k}=l_{i_0,k}$ where i_0 is the nearest phase before phase i where the components of meta-type k are present in that phase. If the PMS functions in an environment without shocks and only the aging only causes failure of the components, then the reliability function of PMSs is obtained as follows

$$\begin{aligned}\bar{H}_a(t) &= \sum_{\ell_{1,1}=0}^{n_{1,1}}\cdots\sum_{\ell_{\gamma(t),\mathbb{K}}=0}^{n_{\gamma(t),\mathbb{K}}}\Phi_{\gamma(t)}(\ell_{1,1},...,\ell_{\gamma(t),\mathbb{K}})Pr\left[\bigcap_{i=1}^{\gamma(t)}\bigcap_{k=1}^{\mathbb{K}}\left(N_a^{[i,k]}(t^\star_{i,k})=n_{i,k}-\ell_{i,k}\right)\right]\\ &= \sum_{\ell_{1,1}=0}^{n_{1,1}}\cdots\sum_{\ell_{\gamma(t),\mathbb{K}}=0}^{n_{\gamma(t),\mathbb{K}}}\Phi_{\gamma(t)}(\ell_{1,1},...,\ell_{\gamma(t),\mathbb{K}})\prod_{i=1}^{\gamma(t)}\prod_{k=1}^{\mathbb{K}}\binom{n_{i,k}}{\ell_{i,k}}(R_{i,k}(t^\star_{i,k}))^{\ell_{i,k}}\\ &\quad\times(1-R_{i,k}(t^\star_{i,k}))^{n_{i,k}-\ell_{i,k}},\end{aligned}\tag{2}$$

where $t^\star_{i,k}=\min\{t,\tau_{i+1}\}-\tau_i+\beta_{i,k}$ and $\beta_{i,k}=\sum_{i'\in I_k}(\tau_{i'+1}-\tau_{i'})$ with I_k as the set of numbers of phases that the components of meta-type k are present in those phases before phase i; see, [4, 1]. Also $\gamma(t)$ denotes the function which shows the phase that the system is in that phase at time t.

Note that $N_a^{[i,k]}(t)$ denotes number of the failed components of meta-type k due to aging in phase i, $i=1,2,...,M$, $k=1,2,...,\mathbb{K}$ and $t\in[\tau_i,\tau_{i+1}]$. If the reliability function is being evaluated at $t>\tau_{i+1}$, then $N_a^{[i,k]}(t)=N_a^{[i,k]}(\tau_{i+1})$. Also $R_{i,k}(t)=1-F_{i,k}(t)$, $i=1,2,...,\gamma(t)$, and

$$F_{i,k}(t)=\frac{F_k(t^*_{i,k})-F_k(\beta_{i,k})}{1-F_k(\beta_{i,k})}$$

where $F_k(.)$ is DF of the lifetimes of components of meta-type k, $k=1,2,...,\mathbb{K}$. Note that if all components of a special meta-type fail in a given phase, the PMS may function with the other type of components. Then, $\ell_{i,k}\geq 0$, for $i=1,2,...,\gamma(t)$, $k=1,2,...,\mathbb{K}$. In what follows, we first evaluate the reliability of the PMS only under external shocks. Finally, an expression is given for the case where both the aging and external shocks may cause of components failure.

2.1 PMS with components only under external shocks

Now, we assume that the components of PMS are absolutely reliable under their aging and only the harmful shocks arriving from a random environment are the only cause of the component's failures. Let $N_s^{[i,k]}(t),\ k=1,2,...,\mathbb{K},\ t\geq 0$ be independent counting processes which describe the number of shocks during $[0,t)$ that cause the failure of components of meta-type k in phase i. Also suppose that each fatal shock destroys, with equal probabilities, one of the components functioning at the time of its occurrence and the shock has no impact on the other working components. It should be noted that in each phase of the mission, the process of occurrence of shocks on each meta-type of components is independent of the process of occurrence of shocks on other meta-type of components. Also, the process of occurrence of shocks in each phase of the mission is independent of the process of occurrence of shocks in other phases of the mission. Let

$$P_r^{[i,k]}(t)=Pr(N_s^{[i,k]}(t)=r),\qquad r=0,1,...,\quad i=1,2,...,\mathbb{M},\quad k=1,2,...,\mathbb{K},$$

where $P_r^{[i,k]}(t)$ denotes the probability that r shocks occur on the components of meta-type k for duration time t in phase i. So the reliability of PMS is evaluated as follows

$$\begin{aligned}\bar{H}_s(t)&=\sum_{\ell_{1,1}=0}^{n_{1,1}}\cdots\sum_{\ell_{\gamma(t),\mathbb{K}}=0}^{n_{\gamma(t),\mathbb{K}}}\Phi_{\gamma(t)}(\ell_{1,1},...,\ell_{\gamma(t),\mathbb{K}})Pr\left[\bigcap_{i=1}^{\gamma(t)}\bigcap_{k=1}^{\mathbb{K}}\left(N_s^{[i,k]}(t_i^\star-\tau_i)=n_{i,k}-\ell_{i,k}\right)\right]\\&=\sum_{\ell_{1,1}=0}^{n_{1,1}}\cdots\sum_{\ell_{\gamma(t),\mathbb{K}}=0}^{n_{\gamma(t),\mathbb{K}}}\Phi_{\gamma(t)}(\ell_{1,1},...,\ell_{\gamma(t),\mathbb{K}})\prod_{i=1}^{\gamma(t)}\left\{\prod_{k\in E}\bar{P}_{n_{i,k}}^{[i,k]}(t_i^\star-\tau_i)\right\}\left\{\prod_{k\in E^c}P_{n_{i,k}-\ell_{i,k}}^{[i,k]}(t_i^\star-\tau_i)\right\},\end{aligned}\tag{3}$$

where $E=\{k;\ell_{i,k}=0\}\subseteq\{1,2,...,\mathbb{K}\}$ and

$$\bar{P}_{n_{i,k}}^{[i,k]}(t_i^\star-\tau_i)=\sum_{u_{i,k}=n_{i,k}}^{\infty}P_{u_{i,k}}^{[i,k]}(t_i^\star-\tau_i).$$

Note that if all components of type k, $k=1,2,...,\mathbb{K}$, fail in phase i, $i=1,2,...,M$, we can conclude that at least $n_{i,k}$ out of $N_s^{[i,k]}(t_i^\star-\tau_i)$ shocks have occurred. Therefore, we have

$$Pr(N_s^{[i,k]}(t_i^\star-\tau_i)=n_{i,k}-\ell_{i,k})=\begin{cases}\bar{P}_{n_{i,k}}^{[i,k]}(t_i^\star-\tau_i)=\sum_{u_{i,k}=n_{i,k}}^{\infty}P_{u_{i,k}}^{[i,k]}(t_i^\star-\tau_i) & \ell_{i,k}=0\\ P_{n_{i,k}-\ell_{i,k}}^{[i,k]}(t_i^\star-\tau_i) & \ell_{i,k}=1,2,...,n_{i,k}.\end{cases}$$

2.2 PMS with components under aging and external shocks

Finally, if both the internal failure and external shocks cause the failure of the components of PMS, we have $N^{[i,k]}(t)=N_a^{[i,k]}(t_{i,k}^\star)+N_s^{[i]}(t_i^\star-\tau_i)$ for $t\geq\tau_i$ where $N^{[i,k]}(t)$ denotes the number of the failed components of meta-type k subject to aging and fatal shocks until time t in phase i. Also, suppose that each shock destroys, with equal probabilities, one of the components functioning at the time of its occurrence and the shock has no impact on the other working components. Under these assumptions, in the next theorem, the reliability function of the PMS is obtained.

Theorem 2.2. *Consider a PMS with the M phases consisting of n binary components of the $\mathbb{K}$ different meta-type. Suppose that the random failure times of components are statistically independent in the same phases and $F_k(.)$ is the DF of random failure time of the components of meta-type k, $k=1,2,\ldots,\mathbb{K}$. Consider in addition to the aging, the components of meta-type k are subject to* *external shocks. The shocks appear based on counting process* $\{\{N_s^{[i,k]}(t), t \geq 0\}$ *where* $N_s^{[i,k]}(t)$ *shows the number of shocks that occur on the components of meta-type* k *during time* $[0,t)$ *in phase* i*. If occurrence of internal failures and the external shocks are independent of each other, the reliability function of PMS at time* t *is gotten as*

$$\begin{aligned}\bar{H}(t) = &\sum_{\ell_{1,1}=0}^{n_{1,1}} \ldots \sum_{\ell_{\gamma(t),\mathbb{K}}=0}^{n_{\gamma(t),\mathbb{K}}} \Phi_{\gamma(t)}(\ell_{1,1}, ..., \ell_{\gamma(t),\mathbb{K}}) \prod_{i=1}^{\gamma(t)} \\ &\times \left\{ \prod_{k \in E} \sum_{s_{i,k}=0}^{n_{i,k}} \binom{n_{i,k}}{s_{i,k}} (F_{i,k}(t^\star_{i,k}))^{s_{i,k}} (1 - F_{i,k}(t^\star_{i,k}))^{n_{i,k}-s_{i,k}} \bar{P}^{[i,k]}_{n_{i,k}-s_{i,k}}(t^\star_i - \tau_i) \right\} \\ &\times \left\{ \prod_{k \in E^c} \sum_{s_{i,k}=0}^{n_{i,k}-\ell_{i,k}} \binom{n_{i,k}}{s_{i,k}} (F_{i,k}(t^\star_{i,k}))^{s_{i,k}} (1 - F_{i,k}(t^\star_{i,k}))^{n_{i,k}-s_{i,k}} P^{[i,k]}_{n_{i,k}-\ell_{i,k}-s_{i,k}}(t^\star_i - \tau_i) \right\}, \end{aligned} \quad (4)$$

where $E = \{k; \ell_{i,k} = 0\} \subseteq \{1, 2, ..., \mathbb{K}\}$,

$$F_{i,k}(t^\star_{i,k}) = 1 - R_{i,k}(t^\star_{i,k}) = 1 - \frac{1 - F_k(t^\star_{i,k})}{1 - F_k(\beta_{i,k})},$$

$P^{[i,k]}_r(t) = Pr(N^{[i,k]}_s(t) = r)$, $r = 0, 1, ...,$, $i = 1, 2, ..., \mathbb{M}$, $k = 1, 2, ..., \mathbb{K}$,

$$\bar{P}^{[i,k]}_{n_{i,k}}(t^\star_i - \tau_i) = \sum_{u_{i,k}=n_{i,k}}^{\infty} P^{[i,k]}_{u_{i,k}}(t^\star_i - \tau_i)$$

and $t^\star_i = \min\{t, \tau_{i+1}\}$, $t^\star_{i,k} = \min\{t, \tau_{i+1}\} - \tau_i + \beta_{i,k}$. $\Phi_{\gamma(t)}(\ell_{1,1}, ..., \ell_{\gamma(t),\mathbb{K}})$ *is also evaluated from equation* (1).

Example 2.3. Consider a PMS with $n = 6$ components and two phases. Suppose that the duration time in each phase is 10 hours. The task in Phase 1 is done if all components A, B, and C work. Phase 2 is also completed if components C, and F work, and one of the components D, and E function. Assume that the components are of three types: (type 1) components A and B with exponential lifetimes with the rate 10^{-3} hours, (type 2) component C with exponential lifetime with the rate 2×10^{-3} hours and (type 3) components D, E and F have exponential lifetimes with the rate 3×10^{-3} hours. Table 1 gives the non-zero elements of the survival signature of the described PMS based on relation (1).

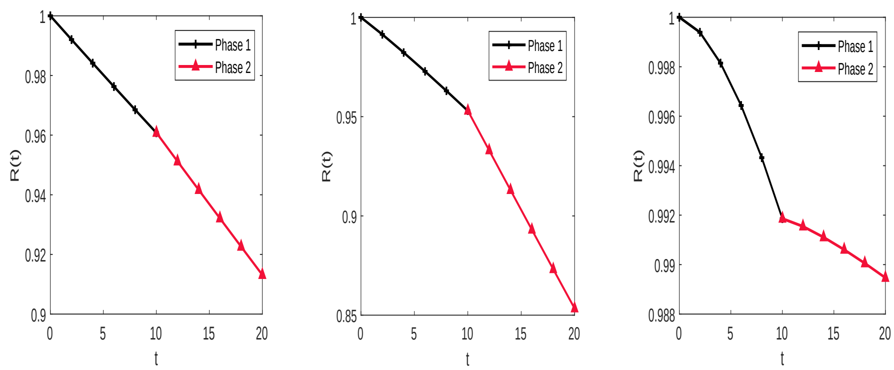
Table 1. Survival signature of the PMS described in Example 2.3.

Phase 1			Phase 1 + 2				
$\ell_{1,1}$	$\ell_{1,2}$	Φ_1	$\ell_{1,1}$	$\ell_{1,2}$	$\ell_{2,2}$	$\ell_{2,3}$	Φ_2
2	1	1	2	1	1	3	1
			2	1	1	2	$\frac{2}{3}$

Assume that the shock processes are based on the non-homogeneous Poisson Processes (NHPP) which affect the PMS components of meta-type k with intensity function $\lambda_{i,k}(t)$ in phase i , $i = 1, 2, 3$. Let $\lambda_{1,1}(t) = 0.00017t^{0.7}$, $\lambda_{1,2}(t) = 0.00015t^{0.5}$, $\lambda_{2,2}(t) = 0.00013t^{0.3}$, and $\lambda_{2,3}(t) = 0.00011t^{0.1}$. The lifetimes of components of meta-type k are assumed to be independent of the process of shock's occurrence. Then,

$$P_r^{[1,1]}(t) = e^{10^{-4}t^{1.7}} \frac{(10^{-4}t^{1.7})^r}{r!},$$
$$P_r^{[1,2]}(t) = e^{10^{-4}t^{1.5}} \frac{(10^{-4}t^{1.5})^r}{r!},$$
$$P_r^{[2,2]}(t) = e^{10^{-4}t^{1.3}} \frac{(10^{-4}t^{1.3})^r}{r!},$$
$$P_r^{[2,3]}(t) = e^{10^{-4}t^{1.1}} \frac{(10^{-4}t^{1.1})^r}{r!}.$$

Figure 1 depicts the plots of $\bar{H}_a(t)$, $\bar{H}_s(t)$ and $\bar{H}(t)$, using relations (2), (6) and (4), respectively. We also compare the reliability of this PMS based on the different proposed models in Figure 2.



(a) $\bar{H}_a(t)$ (b) $\bar{H}(t)$ (c) $\bar{H}_s(t)$

Figure 1: Reliability of the PMS in Example 2.3 based on the proposed models.

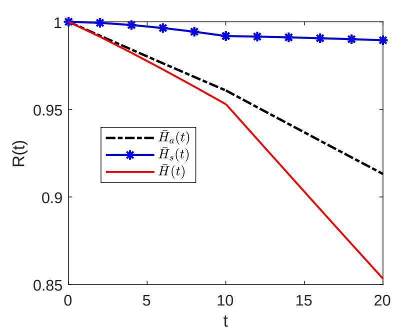


Figure 2: Comparison the reliability of PMS in Example 2.3 based on the proposed models.

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A new discrete-time mixed δ-shock model

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Abstract

In this paper, a mixed δ-shock model with discrete-time is studied by combining δ-shock and extreme shock models. In this model, a system fails in two ways: first, when k interarrival times between two consecutive shocks with magnitude larger than the critical threshold γ are in $[\delta_1, \delta_2], \delta_1 < \delta_2$; and second, when the interarrival time between two consecutive shocks is less than δ_1. The lifetime of the system under the proposed mixed δ-shock model is investigated. Finaly, a numerical example is presented.

Keywords: Discrete time, Interarrival times, Lifetime, Mixed δ-shock model

1 Introduction

A shock model is introduced to represent the operating system failure process. In reliability, four major shock models are studied: i) Shanthikumar and Sumita (1983) [12] and Gut (1990) [6] introduced the cumulative shock model; ii) Gut and Hüsler (1999) [7] studied an extreme shock model that results in the system failure if the magnitude of a shock is more than a threshold γ; iii) the run shock model proposed by Mallor and Omey (2001) [10], and iv) the δ-shock model, a special type of shock model, in which the system fails if the interarrival time between two consecutive shocks is less than a critical threshold δ, and it is studied in detail by Li et al. (1999) [8], Wang and Zhang (2001) [13], Bai and Xiao (2008) [1] and Eryilmaz (2013) [3]. Eryilmaz (2012) [2] studied the life behavior of a system by assuming the arrival shocks to be a type of mixed shock model under the discrete probability

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distribution. Eryilmaz (2015) [4] surveyed three different discrete-time shock models in two ways: (i) shocks are independent, and (ii) shocks are Markov dependent. Lorvand et al. (2020) [9] investigated an extended discrete-time mixed δ-shock model. Eryilmaz and Kan (2021) [5] studied a mixed shock model for the case when the times between successive shocks and the magnitudes of shocks have discrete phase-type distributions.

The purpose of this paper is to discuss the discrete-time version of the introduced mixed δ-shock model by Roozegar et al. (2023) [11]. In this proposed mixed δ-shock model it is assumed that i) the magnitudes of arrival shocks are random, and ii) interarrival times between two consecutive shocks are independent and identically distributed (i.i.d.) sequence of geometric distribution with parameter p. According to the definition, the multi-state system would fail in two ways: first, when k interarrival times between two consecutive shocks with a magnitude larger than the critical threshold γ are in $[\delta_1, \delta_2], \delta_1 < \delta_2$; and second, when the interarrival time between two consecutive shocks is less than δ_1.

The rest of this paper is as following. We investigate the lifetime of the system under this discrete-time mixed δ-shock model in Section 3. In Section 5, an example is established to evaluate the results. Finally, the concluding remarks of this paper are presented in Section 4.

We will use the following terms to examine the properties of the behavior of the lifetime of this mixed δ-shock model:

N	Number of interarrival times between two consecutive shocks until the system fails completely
Z_i	The magnitude of the ith shock
X_i	Interarrival time between the $(i-1)$th and ith shocks, for $i = 1, 2, ...$
δ_j	The critical threshold for δ-shock, $j = 1, 2$
γ	The critical threshold for shock magnitude
k	Number of interarrival times between two consecutive shocks with our considered condition $\delta_1 < X_i < \delta_2, Z_i > \gamma$
T	Lifetime of the system
F	Cumulative distribution function

2 The behavior of the system's lifetime

To obtain the lifetime of this discrete-time mixed δ-shock model, let N denote the number of interarrival times between two successive shocks that cause the system to fail. So, $N = n$ means that n shocks arrived at the system. Then, it can be enumerated as follows, for $j = 0, 1, \ldots, k-1$ and $l = 0, 1, \ldots, n-j-1$:

$$
\begin{aligned}
(N=n)=&\Big[k-1 \; of \; (n-1) \; (X_i,Z_i) \; are \; (\delta_1<X_i<\delta_2, Z_i>\gamma)\\
&\quad and \Big\{ j \; of \; n-k \; (X_i,Z_i) \; are \; (\delta_1<X_i<\delta_2, Z_i<\gamma)\\
&\quad and \; n-k-j \; of \; n-k \; (X_i,Z_i) \; are \; X_i>\delta_2)\Big\}\\
&\quad and \; \delta_1<X_n<\delta_2, Z_n>\gamma\Big]\\
&\cup\Big[k-1 \; of \; (n-1)(X_i,Z_i) \; are \; (\delta_1<X_i<\delta_2, Z_i>\gamma)\\
&\quad and \Big\{ j \; of \; n-k \; (X_i,Z_i) \; are \; (\delta_1<X_i<\delta_2, Z_i<\gamma)\\
&\quad and \; n-k-j \; of \; n-k \; (X_i,Z_i) \; are \; X_i>\delta_2)\Big\} \; and \; X_n<\delta_1\Big]\\
&\cup\Big[j \; of \; (n-1) \; (X_i,Z_i) \; are \; (\delta_1<X_i<\delta_2, Z_i>\gamma)\\
&\quad and \Big\{ l \; of \; n-j-1 \; (X_i,Z_i) \; are \; (\delta_1<X_i<\delta_2, Z_i<\gamma)\\
&\quad and \; (n-j-1-l) \; of \; n-j-1 \; (X_i,Z_i) \; are \; X_i>\delta_2)\Big\} \; and \; X_n<\delta_1\Big].
\end{aligned}
$$

The following Theorem derived the pmf of lifetime of this discrete-time mixed δ-shock model according to the definition of N.

Theorem 2.1. *Suppose X_is are the interarrival times between two consecutive shocks and Z_is are the magnitudes of shocks and these are mutually independent, for $i=1,2,\cdots$. Let $T=\sum_{i=1}^{N} X_i$ be the lifetime of the system. Then, the pmf of the system's lifetime is as follows:*

$$
\begin{aligned}
P(T=n)=&\sum_{i=k+1}^{[\frac{n+(k+1)(\delta_2-\delta_1)}{(\delta_2-\delta_1)+1}]} \binom{i-2}{k-1}\Bigg[\sum_{l=0}^{\min(k,[\frac{n-i-(i-(k+1))(\delta_2-\delta_1)}{(\delta_2-\delta_1)}])} (-1)^l\binom{k}{l}\\
&\times\binom{n-(i-(k-l)-1)(\delta_2-\delta_1)-1}{i-1} p^i(1-p)^{n-i}F^{i-k}(\gamma)\bar{F}^k(\gamma)\\
&+\sum_{j=0}^{k-1}\sum_{i=j+1}^{[\frac{n+(j+1)(\delta_2-\delta_1)}{(\delta_2-\delta_1)+1}]} \binom{i-1}{j}\Bigg[\sum_{l=0}^{\min(j,[\frac{n-i-(i-(j+1))(\delta_2-\delta_1)}{(\delta_2-\delta_1)}])} (-1)^l\binom{j}{l}\\
&\times\binom{n-(i-(j-1)-1)(\delta_2-\delta_1)-1}{i-1} p^i(1-p)^{n-i}F^{i-j-1}(\gamma)\bar{F}^j(\gamma).
\end{aligned}
\tag{1}
$$

3 Computational results

In this section, an example of this study is carried out to validate the analytical results obtained here. It is assumed that the interarrival times $X_1, X_2, \cdots$ and the magnitudes of shocks $Z_1, Z_2, \cdots$ are i.i.d. random variables having the geometric and the exponential distribution with the probability $p=0.8$ and mean 0.5, respectively, and that they are also mutually independent.

Figure 1 presents the pmf of system lifetime $P(T=n)$ for $\delta_1=2$, $\delta_2=4$, $\gamma=0.2$, $\lambda=2$, $p=0.8$ with respect to $k=1,2,3,4$. As can be observed, the system's lifetime decreases when k increasing.

Figure 2 displays the plot of $P(T = n)$ with respect to $\gamma = 0.2, 0.5, 0.8, 1$. As shown, the system's lifetime decreases when γ increasing. Also, the plot of $P(T = n)$ is shown in Figure 3 with respect to $k = 2, 3, 4$ where the system's lifetime decreases when k increasing. In addition, with increasing interarrival time $[\delta_1, \delta_2]$ the lifetime of the system increases.

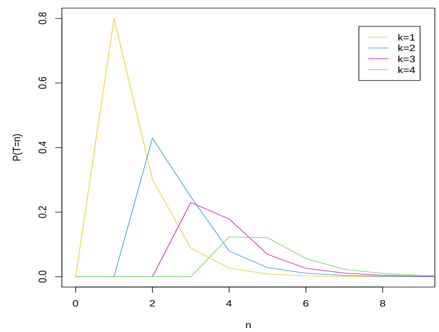

Figure 1: The $P(T = n)$ when $\delta_1 = 2, \delta_2 = 4, \gamma = 0.2, \lambda = 2, p = 0.8$ and for different values of k.

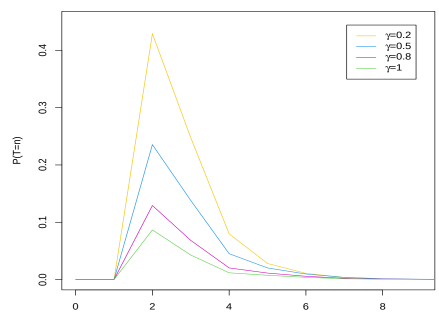

Figure 2: The $P(T = n)$ when $\delta_1 = 2, \delta_2 = 4, \lambda = 2, p = 0.8$ and for different values of γ.

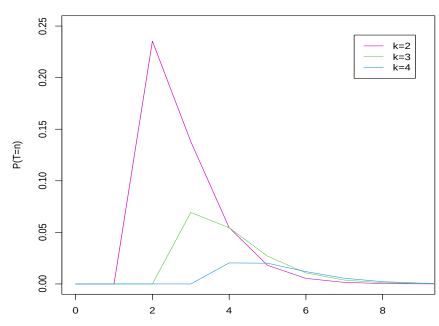

Figure 3: The $P(T = n)$ when $\gamma = 0.2, \lambda = 2, p = 0.8$ and for different values of k.

4 Conclusion

In this study, a mixed δ-shock model with discrete-time is defined by combining δ-shock and extreme shock models, such that it causes the failure of a multi-state system in two ways: first, when k interarrival times between two consecutive shocks with a magnitude of shock larger than threshold γ is in $[\delta_1, \delta_2]$, and second, when the time among two consecutive shocks is less than δ_1. By assuming that the shocks occur independently and randomly with the magnitude Z_i and the interarrival times among two consecutive shocks X_i which are i.i.d. random variables, we have derived explicit expressions for the lifetime of the proposed mixed δ-shock model for system's lifetime.

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System survival probability based on mission abort policy

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Abstract

In this talk, we focus on the importance of system survivability in critical systems, such as satellites and aircraft, where failure can result in significant economic loss and human damage. We propose a mission abort policy as an effective means to enhance system survivability and reduce the risk of system failure. Specifically, we consider a coherent system with n components and abort the system mission if L components fail. Our approach provides insights into the design of mission abort policies that can improve system survivability and reduce economic losses.

Keywords: Mission abort policy, Signature vector, Mission success probability, System survival probability

1 Introduction

The survivability of critical systems takes precedence over completing a mission when failure during the mission can result in significant economic loss and potential harm to human lives and the environment. Examples of critical systems include spaceships, aircraft, drones, satellites, and data processing computer systems, where failure can lead to damage or loss of these objects. A mission abort policy

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can be implemented when a certain malfunction condition occurs or a certain number of components fail, followed by a rescue procedure to save the system and prevent casualties, economic loss, and environmental hazards. The mission success probability is a crucial metric for evaluating the performance of critical systems, representing the probability of successfully completing a mission within a specific time frame. Previous studies by Levitin et al. (2014), Peng et al. (2016), and Liu et al. (2018) have focused on maximizing the mission success probability. Our paper emphasizes the importance of system survivability and proposes a mission abort policy as an effective approach to enhance the survivability of critical systems and reduce potential losses. Previous studies on evaluating mission success probability have not considered the impact of mission abort policy on system survival probability and mission success probability. Traditional models focus solely on mission success probability without accounting for the effectiveness of mission abort policies in enhancing system survivability. Thus, computing mission success probability based on a mission abort policy is not a typical reliability problem. A new approach is needed to accurately evaluate the performance of critical systems and the effectiveness of mission abort policies in reducing potential losses and enhancing system survivability.

In critical systems, completing missions within a specified time frame is crucial. Mission abort policies can enhance system survivability by minimizing the potential for system failures and increasing the system's chance of survival in hazardous environments. This policy can effectively reduce economic losses by improving the system's survival probability and providing greater protection against risk. Additionally, mission abort policies can improve efficiency and reliability, leading to reduced operating costs. Myers' seminal work in 2009 on mission abort policies for a k-out-of-n system with an exponential distribution demonstrated the effectiveness of such policies in reducing overall failure probability compared to systems without abort policies. This work laid the foundation for further investigations into mission abort policies for complex systems with multiple sources of uncertainty. Mission abort policies have proven to enhance the reliability of critical systems and expand their capabilities to various tasks. Levitin et al. (2018) extended the mission abort model to heterogeneous systems and adaptive abort policies, but their model did not consider the dynamic nature of stochastic environments and their impact on system behavior and performance. To overcome this limitation, further investigation is needed to explore the effects of stochasticity on the operation, optimization, and control of adaptive abort policies in heterogeneous systems. Levitin and Finkelstein (2018) investigated the mission abort policy based on the generalized extreme shock model and incorporated the effect of the environment. However, their work did not account for the variability of the environment, which can significantly impact the performance of the mission abort policy in real-world scenarios.

Previous studies have investigated the application of the generalized extreme shock model and the duration of a defective state. Recently, Karimi and Tavangar (2023) extended and analyzed the mission abort policy for a coherent system. In situations where a certain number of components fail, a mission may need to be aborted to reduce the risk of system failure, requiring the immediate activation of a rescue procedure. Samaniego (2007) and Kochar et al. (1990) obtained the reliability function of a coherent system based on the signature vector of the system.

A main tool in studying coherent systems is the concept of signature. Suppose a coherent system consists of n components whose failure times $X_1, X_2, ..., X_n$ are independent and identically distributed with a common continuous distribution $F(t)$, and assume that $X_{1:n}, X_{2:n}, ..., X_{n:n}$ denote the order statistics corresponding to X_i's. The system signature is defined as

$$s_i = P(T = X_{i:n}), \quad i = 1, 2, ..., n, \tag{1}$$

where T denotes the system lifetime. In fact, s_i is the probability that the component with lifetime $X_{i:n}$ causes the system failure. The probability vector $\underline{s} = (s_1, s_2, ..., s_n)$ is then called the signature vector of such a coherent system. It can be verified that $s_i = \frac{n_i}{n!}$, where n_i is the number of ways that distinct $X_1, X_2, \ldots, X_n$ can be ordered so that the ith ordered quantity corresponds to the system failure. Then, the reliability function $\bar{F}_T(t)$ of the system can be represented as

$$\bar{F}_T(t) = P(T > t) = \sum_{i=1}^{n} s_i P(X_{i:n} > t). \tag{2}$$

In this talk, we present a novel age-based system for computing the system survival probability and mission success probability of a coherent system. We propose an extended mission abort policy to provide a detailed analysis of the system's behavior, including the probability of completing the rescue procedure. We also investigate and discuss various stochastic order properties of the rescue procedure.

2 Main results

In critical systems, the survival of the system is paramount, as failure can result in significant economic loss. This paper considers a coherent system with n components that perform a fixed duration t_M(mission duration) under adverse environmental conditions, such as electric shock and lightning. To enhance system survivability and minimize potential losses, Karimi and Tavangar (2023) proposed a mission abort policy that triggers the rescue or recovery procedure if a predetermined number of components (L components) fail then, in order to increase the survival of the system and decrease the loss of the system, the mission must be aborted and immediately must be activated the rescue or recovery procedure. Note that we treat the parameter L as a decision variable, enabling us to optimize the mission abort policy effectively. In the following, some important criteria are developed under the aforementioned assumptions.

2.1 Mission Success Probability (MSP)

The probability of mission success is a critical performance metric for evaluating the reliability of a system, representing the likelihood of completing a specific mission within or before a given deadline. To compute the probability of mission success, one must consider whether the system's total life expectancy exceeds the mission duration t_M and whether the life expectancy of the Lth component is also greater than t_M, where $1 \leq L \leq n-1$. Ensuring that both of these criteria are met can significantly improve system reliability. The mission success probability is defined as

$$\begin{aligned} MSP = R(t_M, L) &= \mathbb{P}(T > t_M, X_{L:n} > t_M) \\ &= \bar{S}_L \mathbb{P}(X_{L:n} > t_M) + \sum_{j=1}^{L} s_j \mathbb{P}(X_{j:n} > t_M) \end{aligned}$$

where $\bar{S}_L = \sum_{j=L+1}^{n} s_j$. It can be verified that the MSP can also be written as (see Karimi and Tavangar, 2023)

$$MSP = \sum_{k=0}^{L-1} \bar{S}_k \binom{n}{k} \big(F(t_M)\big)^k \big(\bar{F}(t_M)\big)^{n-k}. \tag{3}$$

Notice that, as expected, the MSP is decreasing in t_M; that is, the mission success probability of the system declines with increasing the mission duration. Also, the MSP is increasing in $L \in \{1, 2, ..., n-1\}$. This follows easily from the fact that

$$[T > t_M, X_{L:n} > t_M] \subseteq [T > t_M, X_{L+1:n} > t_M].$$

In the next theorem, we compared two coherent n-component systems based on their mission success probabilities.

Theorem 2.1. *(Karimi and Tavangar, 2023) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two coherent systems with respective signature vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. Suppose that F and G are the common distributions of component lifetimes of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. If $\boldsymbol{s}_1 \leq_{\text{st}} \boldsymbol{s}_2$ and $F \leq_{\text{st}} G$, then $MSP^{(1)} \leq MSP^{(2)}$, where $MSP^{(j)}$ is the mission success probability corresponding to system $\mathcal{S}_j$, $j = 1, 2$.*

2.2 Rescue Procedure (RP)

The rescue protocol is activated only when the Lth component of the system fails at an unforeseeable time t_F (the time at which mission abort occurs) before the mission termination time t_M. For a coherent system, the system does not fail during this period if the difference between the system's lifetime and the failure time of the Lth component exceeds a fixed time t_R (duration of recovery or rescue period), provided that the system's lifetime exceeds that of the Lth component and the Lth order statistic should be less than t_M. Specifically, the system's lifetime must be greater than or equal to $t_F + t_R$. It is important to note that the initiation and duration of the rescue procedure cannot be predetermined before the mission starts, emphasizing the need for an adaptive mission abort policy that can effectively improve system survivability and reduce potential losses. The RP is defined as

$$\begin{aligned} RP &= \mathbb{P}(T - X_{L:n} > t_R \mid T > X_{L:n}, X_{L:n} < t_M) \\ &= \mathbb{P}\big(T - X_{L:n} > t_R \mid X_{L:n} < min(T, t_M)\big) \end{aligned}$$

2.3 System survival probability (SSP)

The system survival probability is the probability that the mission and its associated rescue procedure will be completed successfully. In other words, it captures the probability of success for a given mission and any related rescue protocols.

$$\begin{aligned} SSP &= S(t_M, L, t_R) \\ &= \mathbb{P}[(T > t_M, X_{L:n} > t_M) \cup (T - X_{L:n} > t_R, X_{L:n} < min(T, t_M))] \\ &= MSP + RP \times \mathbb{P}[X_{L:n} < min(T, t_M)], \end{aligned}$$

Where the second term represents the probability that the rescue procedure can successfully save the system. The formula for the computation of RP and SSP along with their basic properties can be found in Karimi and Tavangar (2023).

3 Examples

In this section, we provide some examples to examine the theoretical results of previous sections. Suppose there are two bridge systems connected in parallel, and each system consists of components that follow a Weibull distribution with shape parameter 1 and scale parameter 1 (the exponential distribution). The signature vector for this parallel system can be calculated as follows:

$$s = (0, 4/45, 19/90, 3/10, 86/315, 34/315, 2/105, 0, 0, 0)$$

We have also determined the mission success probability for this system, which is plotted in Figure 1. If the mission abort occurs at the first order statistic (i.e., the first component failure), the probability of mission success is the lowest. This is because a mission abort at this early stage indicates a critical failure that may have a significant impact on the success of the mission. On the other hand, as the order statistic for mission abort increases (i.e., the mission can continue despite more components failing), the probability of completing the mission also increases. This is because the system is still functioning despite the loss of one or more components, which indicates a higher level of redundancy and resilience in the system

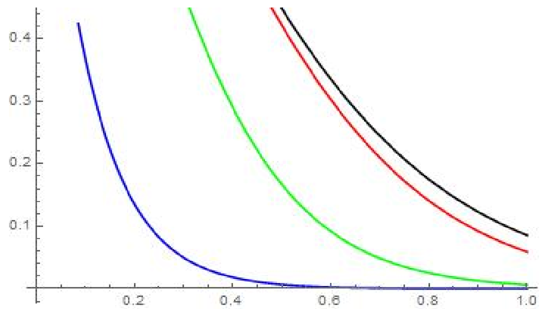

Figure 1: Mission success probability for $L = 1$ (Blue), $L = 3$ (Green), $L = 5$ (Red), $L = 7$ (Black).

We also determined the probability of completing the rescue procedure for different scenarios where the mission abort occurs at different order statistics, as shown in Figure 2. As expected, the probability of completing the rescue method is higher when the mission abort occurs at the first order statistic, indicating a critical failure that has a significant impact on the rescue procedure's success. Conversely, if the mission abort occurs at a higher order statistic, the probability of completing the rescue method decreases. This is because the system has experienced multiple failures, indicating a lower level of redundancy and resilience, which can increase the risk of further failures and reduce the likelihood of completing the rescue procedure successfully. Understanding the impact of order statistics on the probability of completing the rescue procedure is crucial in designing reliable and robust systems for emergency situations. By considering the order statistics and designing systems with redundancy and resilience, we can improve the probability of mission success and ensure the safety and security of the rescue operation and its personnel.

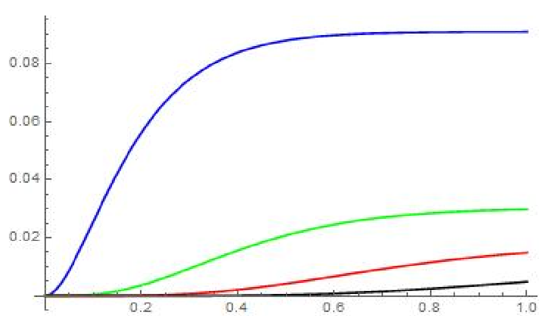

Figure 2: Rescue procedure for $L = 1$ (Blue), $L = 3$ (Green), $L = 5$ (Red), $L = 7$ (Black).

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Emprical estimators for multi-component conditional stress-strength parameter

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Abstract

In many of reliability models, there exist certain information about the strength and stresses that experienced by the system. We are interested in how the model functions via these extra information or whether employing them does improve the performance of the system. In the present study the conditional stress-strength parameter have been investigated for s of k systems and the multi-component conditional stress-strength parameter (MCCSSP) has been estimated by using the Bayesian and non-parametric methods. In the case of having extra information about the parameters of the system, a closed form has been derived for the Bayes estimator of MCCSSP and has been calculated by using an algorithm together with Monte Carlo method. For simplicity, it has been done under the assumption of exponential distributions for the strength and stress random variables and gamma conjugates. For the case of non-exponential or general stress or strengths, the nonparametric estimator of the considered parameter has been derived. Finally to verify the analytic results, some simulation study for the Bayes estimator as well as nonparametric estimation of a real data set and some comparisons have been done.

Keywords: Conditional Reliability, Multi-Component Systems, Stress-Strength Parameter, Nonparametric Estimator, Bayes Estimator.

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1 Introduction

The effects of resistance and shocks which enter to a system are usually studied via a stress-strength model. Due to the application of this model in various fields of science and technologies, recently they have been studied a lot. In such models, when the stress that experienced by the system have been represented by a random variable (RV) X and its strength by the RV Y, the stress-strength parameter is denoted by $R = P(X > Y)$, it measures the chance of the system failure. For the majority of the well-known distributions, including Normal, Exponential, Pareto, Uniform, Weibull, Gamma, Beta, logistic, and Laplace, R has been studied by [3].

There are situations that one have some information about the stress and strength RV's and knows that they are greater than some pre-specified values, or one wants to know how much a system can be reliable when stress and strength increase or decrease. In view of such conditions, the conditional stress-strength parameter was introduced by [4] as:

$$R^{|a,b} = P(X > Y \mid X > a, Y > b). \tag{1}$$

The reliability of a multi-component stress-strength model was first developed by [1]. In this article, we have focussed on the non-parametric and Bayes estimation of conditional stress-strength parameter in multi-component systems and employ certain available information. The introduction of MCCSSP as well as many other of its properties and behaviors can be found in [2].

The structure of this article is as follows: Some notation and formulas concerning MCCSSP will have been provided in Section 2. In Section 3, the Bayes estimator of this parameter has been obtained by adopting an algorithm and by using the Monte Carlo method. The corresponding nonparametric estimator of this parameter has been obtained in Section 4. Section 5 is devoted to the presentation of some simulation studies on the Bayesian and nonparametric estimators and their comparison. Some numerical results for a real data-set have been presented in Section 6. Finally in Section 7, some concluding remarks have been given.

2 The MCCSSP

The results of this section can be found in [2], where the MCCSSP will have been defined together with a general formula for its computation.

Definition 2.1. Consider the independent RV's $X_1, ..., X_k$ with common continuous distribution function $F(\cdot)$, independent of continuous RV Y with distribution function $G(\cdot)$. The MCCSSP is defined as:

$$R_{s,k}^{|a,b} = P(\text{at least s of } X_1, ..., X_k \text{ exceed } Y \mid X_1 > a, ..., X_k > a, Y > b). \tag{2}$$

A formula for computing (2) has been presented in the following theorem.

Theorem 2.2. *If $R_{a,b}^{|s,k}$ is defined by* (2)*, then*

$$R_{s,k}^{|a,b} = \begin{cases} \frac{\sum_{i=s}^{k} \binom{k}{i} \int_b^\infty [1-F(y)]^i [F(y)-F(b)]^{k-i} dG(y)}{[1-F(a)]^k [1-G(b)]} & a \leq b \\ \frac{\sum_{i=s}^{k} \binom{k}{i} (\int_a^\infty [G(x)-G(b)] dF(x))^i (\int_a^\infty [1-G(x)] dF(x))^{k-i}}{[1-F(a)]^k [1-G(b)]} & a > b \end{cases} \tag{3}$$

According to the calculations resulting in the formula (3), if $X_1, \ldots, X_k$ have different distributions, it is not easy to calculate the analogous of this formula. In practice, there are some situations in them X_i and Y have the same distributions, so in what follows, the formula (3) has been presented when $X_1, \ldots, X_k$ and Y have the same distributions.

Corollary 2.3. *Suppose that the continuous RV's $X_1, ..., X_k$ and Y are independent and identically distributed with probability density function(pdf) $f(.)$ and cumulative distribution function(cdf) $F(.)$. Then,*

$$R_{s,k}^{|a,b} = \begin{cases} \frac{\sum_{i=s}^{k} \binom{k}{i} \int_{F(b)}^{1} [1-y]^i [y-F(b)]^{k-i} dy}{[1-F(a)]^k [1-F(b)]}, & a \leq b \\ (\frac{1}{2})^k \frac{\sum_{i=s}^{k} \binom{k}{i} [1-2F(b)+F(a)]^i [1-F(a)]^{k-i}}{[1-F(b)]}, & a > b. \end{cases} \tag{4}$$

The exponential RV is the most exploited lifetime distribution, so in the sake of simplicity and wide applications the measure (3) has been evaluated for the exponentially distributed stresses and strength RV's with different parameters. The probability density and cumulative distribution functions of a random variable $X \sim E(\alpha)$ are denoted by: $f(x) = \alpha e^{-\alpha x}$, and $F(x) = 1 - e^{-\alpha x}$ where $x \geq 0, \alpha > 0$.

Corollary 2.4. *Suppose that $X_i \sim E(\lambda_1)$ for $i = 1, \ldots, k$ and $Y \sim E(\lambda_2)$ are independent, we have:*

$$R_{s,k}^{|a,b} = \begin{cases} \lambda_2 e^{-\lambda_1 k(b-a)} \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i,j} \frac{(-1)^j}{\lambda_1(i+j)+\lambda_2} & a \leq b \\ e^{-\lambda_2(ak-b)} [\frac{\lambda_1}{\lambda_1+\lambda_2}]^k \sum_{i=s}^{k} \binom{k}{i} [\frac{\lambda_1+\lambda_2}{\lambda_1} e^{-\lambda_2(b-a)} - 1]^i & a > b. \end{cases} \tag{5}$$

3 Bayes Estimation

In this section, the Bayesian estimation of the reliability parameter (5) has been considered. Suppose that the parameters λ_1 and λ_2 are RV's, and have independent Gamma prior distributions with parameters $(\alpha_i, \beta_i), i = 1, 2$ respectively. The pdf of a random variable $X \sim Gamma(\alpha_i, \beta_i)$ is denoted by:

$$\pi(x) = \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-\beta_i x} \qquad x > 0,\ \alpha_i > 0,\ \beta_i > 0.$$

The joint posterior density function of the parameters based on this prior density and the likelihood function can be written as follows:

$$\pi^*(\lambda_1, \lambda_2 \mid \mathbf{x}, \mathbf{y}) = \frac{\pi(\lambda_1, \lambda_2, \mathbf{x}, \mathbf{y})}{\int_0^\infty \int_0^\infty \pi(\lambda_1, \lambda_2, \mathbf{x}, \mathbf{y}) d\lambda_1 d\lambda_2}$$

where

$$\begin{aligned} \pi(\lambda_1, \lambda_2, \mathbf{x}, \mathbf{y}) &= \pi(\lambda_1)\pi(\lambda_2)L(\lambda_1, \lambda_2) \\ &\propto \lambda_1^{\alpha_1+n-1} e^{-\lambda_1(\beta_1 + \sum_{i=1}^{n} x_i)} \lambda_2^{\alpha_2+m-1} e^{-\lambda_2(\beta_2 + \sum_{j=1}^{m} y_j)}. \end{aligned}$$

After some calculus, one can see that the posterior density functions of λ_1 and λ_2 respectively are as:

$$\pi^*(\lambda_1 | \lambda_2, \mathbf{x}, \mathbf{y}) \propto \Gamma(\alpha_1 + n, \beta_1 + \sum_{i=1}^{n} x_i),$$

$$\pi^*(\lambda_2 | \lambda_1, \mathbf{x}, \mathbf{y}) \propto \Gamma(\alpha_2 + m, \beta_2 + \sum_{j=1}^{m} y_j).$$

The Bayes estimator of $R_{s,k}^{|a,b}$ under the squared error loss (SEL) is obtained as:

$$\tilde{R}_{s,k}^{|a,b} = E(R_{s,k}^{|a,b} | \mathbf{x}, \mathbf{y}) = \int_0^\infty \int_0^\infty R_{s,k}^{|a,b} \pi^*(\lambda_1, \lambda_2 \mid \mathbf{x}, \mathbf{y}) d\lambda_1 d\lambda_2. \tag{6}$$

It is not possible to calculate equation (6) analytically. Therefore, to compute the Bayes estimate of reliability parameter $R_{s,k}^{|a,b}$, a Monte Carlo (MC) method has been adopted as follows:

Step 1: Set l=1.
Step 2: Generate $X_1, \ldots, X_n$ from $Exp(\lambda_1)$.
Step 3: Generate $Y_1, \ldots, Y_m$ from $Exp(\lambda_2)$.
Step 4: Generate λ_1^l from $Gamma(\alpha_1 + n, \beta_1 + \sum_{i=1}^n x_i)$.
Step 5: Generate λ_2^l from $Gamma(\alpha_2 + m, \beta_2 + \sum_{j=1}^m y_j)$.
Step 6: Compute $R_{s,k}^{l|a,b}$ at $(\lambda_1^l, \lambda_2^l)$.
Step 7: l=l+1.
Step 8: Repeat Steps 2 to 7, M times and obtain the posterior sample $R_{s,k}^{l|a,b}$ for $l = 1, ..., M$.

Now the Bayes estimate of $R_{s,k}^{|a,b}$ with respect to SEL will be obtained as follows:

$$\tilde{R}_{s,k}^{|a,b} = \frac{1}{M} \sum_{l=1}^{M} R_{s,k}^{l|a,b}. \tag{7}$$

4 Nonparametric Estimation

In this section a nonparametric method for estimating $R_{s,k}^{|a,b}$ has been presented. Let $n(.)$ be the counting measure. For the sample space $\mathbf{S}$ and the event $\mathbf{D}$ as a subset of $\mathbf{S}$ the nonparametric estimator of $P(\mathbf{D})$ is defined as $\hat{P}(\mathbf{D}) = \frac{n(\mathbf{D})}{n(\mathbf{S})}$. To obtain the nonparametric estimator of MCCSSP, one may write (2) in the form:

$$R_{s,k}^{|a,b} = \frac{P(\text{at least } s \text{ of } X_1, \ldots, X_k \text{ exceed } Y,\ X_1 > a, \ldots, X_k > a,\ Y > b)}{P(X_1 > a, \ldots, X_k > a)P(Y > b)}, \tag{8}$$

where $P(X_1 > a, \ldots, X_k > a)P(Y > b) > 0$.
Let $\mathbf{A} = \{(x_1, \ldots, x_k, y) \mid \text{at least s of } x_1, \ldots, x_k \text{ exceed } y,\ x_1 > a, \ldots, x_k > a,\ y > b\}$, $\mathbf{B} = \{(x_1, \ldots, x_k) \mid x_1 > a, \ldots, x_k > a\}$ and $\mathbf{C} = \{y \mid y > b\}$. The nonparametric estimator of (8) can be written as follows:

$$R_{s,k}^{NP|a,b} = \frac{n(\mathbf{A})}{n(\mathbf{B})n(\mathbf{C})}. \tag{9}$$

Let $X_{1i}, ..., X_{ki} \sim X$ for $i = 1, ..., n$ and $Y_1, ..., Y_m \sim Y$ be independent random samples. Also, let $I(E)$ be the indicator function of the event E. We have:

$$n(\mathbf{B}) = \sum_{i=1}^{n} \mathrm{I}(X_{1i} > a, \ldots, X_{ki} > a), \tag{10}$$

$$n(\mathbf{C}) = \sum_{j=1}^{m} \mathrm{I}(Y_j > b), \tag{11}$$

and by the properties of the indicator function:

$$\begin{aligned} n(\mathbf{A}) = &\sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{I}(\text{s of } X_{1i}, ..., X_{ki} \text{ exceed } Y_j) \mathrm{I}(X_{1i} > a, \ldots, X_{ki} > a) \mathrm{I}(Y_j > b) + \ldots \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{I}(\text{k of } X_{1i}, ..., X_{ki} \text{ exceed } Y_j) \mathrm{I}(X_{1i} > a, \ldots, X_{ki} > a) \mathrm{I}(Y_j > b). \end{aligned} \tag{12}$$

Let $\mathbf{X}_i = (X_{1i}, \ldots, X_{ki})$ for $i = 1, \ldots, n$. Those observations $\mathbf{X}_i$ and Y_j for them both $\mathbf{X}_i \leq a$ and $Y_j \leq b$ simultaneously, have been removed in calculating $n(\mathbf{A})$, since in details of calculating $P(\mathbf{A})$ or $R_{s,k}^{NP|a,b} = \frac{n(\mathbf{A})}{n(\mathbf{B})n(\mathbf{C})}$, the numerator is an strict subset of denominator. Note that in this case the values of the second and third indicators will automatically equal one in $n(\mathbf{A})$, (12). It is worth noting that the number of remained samples of $\mathbf{X}_i$ and Y_j are $n(\mathbf{B})$ and $n(\mathbf{C})$, so $n(\mathbf{A})$ can be written as follows:

$$\begin{aligned} n(\mathbf{A}) &= \sum_{i=1}^{n(\mathbf{B})} \sum_{j=1}^{n(\mathbf{C})} \mathrm{I}(\text{s of } X_{1i}, ..., X_{ki} \text{ exceed } Y_j) + \ldots \\ &+ \sum_{i=1}^{n(\mathbf{B})} \sum_{j=1}^{n(\mathbf{C})} \mathrm{I}(\text{k of } X_{1i}, ..., X_{ki} \text{ exceed } Y_j). \end{aligned}$$

In the case of $n = m$, the formula (9) may have simpler form and computations, since we only keep those $(X_{1i}, ..., X_{ki}, Y_i)$ $i = 1, ..., n$ which for them $(X_{1i} > a, ..., X_{ki} > a, Y_i > b)$ and remove the rest and also $n(\mathbf{B}) = n(\mathbf{C})$.

In what follows, we introduce a definition and representation for non-parametric estimator of multi-component stress-strength parameter. To the best of our knowledge, interestingly this estimator has not been defined till now.

Definition 4.1. The nonparametric estimator of $R_{s,k}$ is defined as follows:

$$R_{s,k}^{NP} = \frac{n(\mathbf{A})}{n(\mathbf{B})n(\mathbf{C})} \tag{13}$$

where $n(\mathbf{B}) = n$, $n(\mathbf{C}) = m$ and

$$n(\mathbf{A}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{I}(\text{s of } X_{1i}, ..., X_{ki} \text{ exceed } Y_j) + \cdots + \sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{I}(\text{k of } X_{1i}, ..., X_{ki} \text{ exceed } Y_j).$$

Note that (13) can be obtained from (9) by assuming $a = b = 0$.
In applications, the data observed for different stresses may differ greatly in their values. Therefore, selecting a minimum value of a, w.r.t. it all stresses in MCCSSP through definition 1, satisfy the corresponding condition $X_i > a$, may be not useful. So, in what follows, the MCCSSP has been defined in some general way to be more realistic and applicable.

Definition 4.2. The generalized conditional multi-component stress-strength parameter is defined as follows:

$$R_{s,k}^{|a_1,...,a_k,b} = P(\text{at least s of } X_1, ..., X_k \text{ exceed } Y \mid X_1 > a_1, ..., X_k > a_k, Y > b) \tag{14}$$

where the RV's $Y, X_1, ..., X_k$ are independent, $G(\cdot)$ is the continuous distribution function of Y and $F(\cdot)$ is the common continuous distribution function of $X_1, ..., X_k$.

Theorem 4.3. *If $X_{ri} > max(a_1, \ldots, a_k)$ for $r = 1, \ldots, k$; $i = 1, \ldots, n$ and $Y_j > b$ for $j = 1, \ldots, m$ then $R_{s,k}^{NP|a_1,...,a_k,b} = R_{s,k}^{NP}$.*

Proof. Replace $\mathrm{I}(X_{1i} > a_1, \ldots, X_{ki} > a_k)$ with $\mathrm{I}(X_{1i} > a, \ldots, X_{ki} > a)$ in (10) and (12). Since $\mathrm{I}(X_{1i} > a_1, \ldots, X_{ki} > a_k) = 1$ and $\mathrm{I}(Y_j > b) = 1$ we have $n(\mathbf{B}) = n$, $n(\mathbf{C}) = m$ and $n(\mathbf{A}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{I}(\text{s of } X_{1i}, ..., X_{ki} \text{ exceed } Y_j) + \cdots + \sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{I}(\text{k of } X_{1i}, ..., X_{ki} \text{ exceed } Y_j)$. □

Of course, (2) is a special case of (14). In parametric case (see [2] for the MLE method) when $a_1, \ldots, a_k$ are closed in values, a can be considered as the minimum or maximum of $a_1, \ldots, a_k$ and hence one may approximate (14) through (3). In some situations, $a_1, \ldots, a_k$ are very different, and using (14) is not very helpful or may not be accurate. In these cases, the non-parametric method is more practical and it is enough to consider $\mathbf{A} = \{(x_1, \ldots, x_k, y) \mid \text{at least s of } x_1, \ldots, x_k \text{ exceed y}, x_1 > a_1, \ldots, x_k > a_k, y > b\}$, and $\mathbf{B} = \{(x_1, \ldots, x_k) \mid x_1 > a_1, \ldots, x_k > a_k\}$ in (9). It is easy to see that the results of nonparametric estimation of (8) can also be used for nonparametric estimation of (14), where a_i is substituted instead of a for $i = 1, \ldots, k$. Note that in this case, one advantage of the nonparametric method is that the assumption of common distribution for stress RV's may be relaxed. The later makes this method much more practical. If $\mathbf{B} = \{(x_1, \ldots, x_k, y) \mid x_1 > a_1, \ldots, x_k > a_k, y > b\}$, then the nonparametric estimator of the generalized MCCSSP where stresses and strength RV's are not independent, can also be easily computed through the same method.

5 Simulation

In this section, a simulation study has been done to assess the quality and the efficiency of performance of $R_{s,k}^{|ab}$, its Bayes and nonparametric estimators. The simulations have been only done for $a \neq b$ since for $a = b$ the conditional and unconditional cases have the same results.
A comparison among $R_{1,3}^{NP|a,b}$ and $\tilde{R}_{1,3}^{|a,b}$ assuming $\alpha_1 = 2$, $\beta_1 = 3$, $\alpha_2 = 5$, $\beta_2 = 4$ for different values of a and b, $n = m = 100$, $\lambda_1 = 0.0003$ and $\lambda_2 = 0.0005$ has been done and the results presented in Tables 1 and 2.

Table 1. Comparison of $R_{1,3}^{NP|a,b}$ and $\tilde{R}_{1,3}^{|a,b}$ for $a \leq b$

a	10	25	70	78	170	215	300
b	20	40	74	120	190	260	310
$R_{1,3}^{\|a,b}$	0.8607	0.8568	0.8653	0.8362	0.8530	0.8340	0.8607
$\tilde{R}_{1,3}^{\|a,b}$	0.8598	0.8559	0.8646	0.8349	0.8519	0.8321	0.8598
$R_{1,3}^{NP\|a,b}$	0.8657	0.8655	0.8697	0.8681	0.8691	0.8646	0.8668
Bias($\tilde{R}_{1,3}^{\|a,b}$)	-0.0008	-0.0009	-0.0007	-0.0013	-0.0010	-0.0019	-0.0008
Bias($R_{1,3}^{NP\|a,b}$)	0.0049	0.0087	0.0043	0.0618	0.0161	0.0305	0.0061

Table 2. Comparison of $R_{1,3}^{NP|a,b}$ and $\tilde{R}_{1,3}^{|a,b}$ for $a > b$

a	7	22	45	67	100	120	240
b	4	11	38	65	90	70	230
$R_{1,3}^{\|a,b}$	0.9437	0.9377	0.9124	0.8877	0.8664	0.8867	0.7532
$\tilde{R}_{1,3}^{\|a,b}$	0.9372	0.9313	0.90559	0.8812	0.8598	0.8805	0.7466
$R_{1,3}^{NP\|a,b}$	0.8654	0.8658	0.8658	0.8653	0.8674	0.8686	0.8680
Bias($\tilde{R}_{1,3}^{\|a,b}$)	-0.0064	-0.0063	-0.0064	-0.0065	-0.0065	-0.0062	-0.0065
Bias($R_{1,3}^{NP\|a,b}$)	-0.0782	-0.0719	-0.0466	-0.0224	0.0097	-0.0181	0.1147

6 Real Data Analysis

In this section the numerical results of the parameters estimation for a real data set with Exponential distribution have been presented. This data set was used for the first time by [5] and can be find in it. These data present the tensile properties of the jute fibres at different gauge lengths 5, 10, 15 and 20 mm which measured in MPa. The data sets corresponding to the breaking strength of jute fibres with 10mm and 15mm gauge lengths have been considered as the stresses measurement and 20mm in gauge lengths, which represents the strength measurement.

Each data has been separately fitted to the some exponential distribution and examined by using the Kolmogorov-Smirnov goodness-of-fit test, the results have been reported in Table 3. The Kolmogorov-Smirnov statistics and the corresponding P-values indicate that the Exponential distribution fits the data sets. The estimation of MCCSSP for different values of a and b by nonparametric methods and Bayesian approach assuming $\alpha_1 = 2$, $\beta_1 = 3$, $\alpha_2 = 5$, $\beta_2 = 4$ for parameters of prior distributions have been presented in Table 4.

The data set consisting of the breaking strength of jute fiber 5 mm in gauge length have been fitted with the Normal distribution with mean 384.37 and standard deviation 188.77 using the Kolmogorov-Smirnov goodness-of-fit test. For this data, the Lilliforce significance correction criteria (modified Kolmogorov-Smirnov test to check the normality of the data) and the P-value are 0.143 and 0.122. Note that by adding this length to the model, the assumption of exponentially for all stresses fails. The nonparametric estimators of MCCSSP for real data and different values of $a_1,\ a_2,\ a_3$ and b have been presented in Table 5 where X_1 has Normal distribution, X_2 and X_3 have exponential distribution.

Table 3. K-S test for strength of jute fiber data

data	Mean	$\hat{\lambda}$	K-S	p-value
10 mm	365.72	0.0027	0.958	0.317
15 mm	367.87	0.0027	0.999	0.271
20 mm	340.74	0.0029	0.727	0.666

Table 4. Values of estimates of MCCSSP for real data

a	30	45	45	78	85	100	220
b	25	50	40	90	75	80	245
$\hat{R}_{1,2}^{\|a,b}$	0.7200	0.6680	0.6893	0.6432	0.6280	0.6288	0.5996
$\tilde{R}_{1,2}^{\|a,b}$	0.7431	0.6737	0.6458	0.6207	0.6477	0.5970	0.6151
$R_{1,2}^{NP\|a,b}$	0.6744	0.6760	0.6886	0.6462	0.6485	0.6485	0.6944

Table 5. Values of $R_{1,3}^{NP|a_1,a_2,a_3,b}$ for real data

a_1	10	42	80	111	150	215	300
a_2	30	58	90	121	160	221	400
a_3	60	71	100	171	170	240	100
b	34	54	85	154	165	220	340
$R_{1,3}^{NP\|a_1,a_2,a_3,b}$	0.730	0.736	0.705	0.750	0.859	0.625	0.750

7 Conclusion

The MCCSSP ($R_{s,k}^{|a,b}$) as an appropriate extension of multi-component stress-strength parameter has been estimated by Bayesian and non-parametric methods. Certain formulas for estimating the MCC-SSP by Bayesian and nonparametric methods has also been presented. Some numerical computations and simulation studies have been done for illustrating the inferential procedures.

In the past decades, a lot of researches have been done for studying the behavior of reliability function in multi-component stress-strength models, many of similar works can be done for the conditional case. As an specific idea, $R_{s,k}^{|a,b}$ can be obtained and estimated for other distributions. As another idea, one may be interested in the amounts of information which are measurable, lost, unpredictable, etc.

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Survival analysis in the presence of competing risks: a Lunn–McNeil approach

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Abstract

This study considers survival data in which each subject can experience only one of several different types of events over follow-up. When only one of several different types of events can occur, we refer to the probabilities of these events as competing risks. This study aimed to model the survival of patients with Brain stroke in the presence of competing risks.

Materials and Methods: This retrospect survival study was conducted on 332 patients with a definitive diagnosis of Brain stroke. The duration of data collection was from June 2005 up to June 2022 and the follow-up period of patients from the time of diagnosis was 17 years. Data were analyzed by the Lunn-McNeil approach at alpha=0.1 with using STATA version 17 (StataCorp, College Station, TX, USA).

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Results: Significant differences were found between Brain stroke and other causes of death for the age category of 69-75 years, Employment status, Family history of stroke, a history of high blood pressure.
Conclusion: Brain stroke -specific and non- Brain stroke -specific mortality had different risk factors. These findings could be utilized to prescribe optimal and specific treatment.

Keywords: Brain Stroke, survival, competing risks, Lunn-McNeil model.

1 Introduction

In some mortality studies, several causes of death have consistently been found, only one of which is the real cause of death. for example, Death in patients with Brain stroke (BS) may occur due to stroke or other causes. This raises the question, what is the leading cause of death in these patients? Therefore, it is necessary to investigate mortality in patients with BS according to the presence of competing risks [13]. In the presence of competing risks, conventional statistical methods cannot be used because simple survival analysis (Cox model) censors deaths due to other causes, leading to biased estimates [14]. Therefore, to better understand the causes of death in patients with BS, we investigated the causes of death in 2 groups of BS patients (death from stroke and death from other causes) by modeling competing risk with the Lunn-McNeill (L-M) approach. BS is the second most common cause of mortality in the world after heart disease (3). In the United States, 795,000 people have strokes each year affecting the brain [5, 6]. The annual number of strokes is forecast to rise to 3.4 million between 2012 and 2030 [5, 8]. It is also estimated that the cost of treating this disease is 34 billion dollar annually [5, 9].
Studying modifiable risk factors can provide insights into appropriate treatment and prevention measures to enhance the survival of patients with BS [8, 10, 11]. The risk factors of BS are divided into 2 groups: the more important risk factors are age, high blood pressure, and heart disease, while second-degree risk factors that are less important include a family history, high blood lipid levels, obesity, smoking, neurological stress, and a history of BS [7]; some of these factors are modifiable, while others are non-modifiable [2]. Understanding the distribution of these factors is of particular importance.

1.1 Statistical modeling:

Data were summarized and reported as frequency and percentage for categorical variables and as mean (standard deviation [SD]) or median for continuous variables. The survival time of patients was calculated by month. Significant risk factors ($p < 0.1$) in the univariate analysis were entered into the multivariate analysis. The L-M model was used to generate results for the cause-specific assessment of death by modifying the Cox proportional hazard (PH) model [12]. Based on the selected model, and for each of the factors entered in the model, an unadjusted hazard ratio (HR) and an adjusted HR (AHR) were presented along with their 90 % confidence intervals. The interaction L-M model was used to compare BS and other causes of death. The Schoenfeld residual test was used to assess the PH assumption for each variable. All analyses were performed using STATA version 17 (StataCorp, College Station, TX, USA).

2 Main results

The median follow-up time was 160.84 months (range, 59.51-164.58 months). Of the BS patients, 227 (68.4%) died from BS, while 54 (16.3%) died from other causes. (Table 1) provide more details on the demographic characteristics.

Table 1. Participants demographic and clinical characteristics for risk factors of death from brain stroke and other causes

Characteristic	N (%)	Brain stroke N (%)	Other causes N (%)
Age category (years)			
≤ 58	89 (26.8)	40(44.94)	12 (13.48)
59- 68	83 (25.0)	54(65.06)	19 (22.89)
69-75	97 (29.2)	81 (83.51)	12 (12.37)
≥ 76	63 (19.0)	52 (82.54)	11 (17.46)
Sex (female)	164 (49.4)	104 (63.41)	28 (17.07)
Employment status (employed)	291 (87.7)	188 (64.60)	52 (17.87)
Education level ($\geq$ diploma)	84 (25.3)	52 (61.90)	12 (14.29)
Place of residence (city)	201 (60.5)	137(68.16)	30 (14.93)
Family history of stroke	80 (24.1)	46 (57.50)	17(21.25)
Family history of heart attack	24 (7.2)	16 (66.67)	2 (8.33)
Heart disease	86 (25.9)	63 (73.26)	12 (13.95)
History of diabetes	60 (18.1)	45 (75.0)	9 (15.0)
History of blood pressure	197 (59.3)	148 (75.13)	23 (11.68)
History of high cholesterol	62 (18.7)	37 (59.68)	10 (16.13)
Smoking (yes)	64 (19.3)	42 (65.63)	12 (18.57)
Water pipe smoking (yes)	11 (3.3)	10 (90.91)	0 (0.0)
Past smoking (yes)	95 (28.6)	69 (72.63)	17 (17.89)
Passive smoking (yes)	59 (17.8)	40 (67.80)	8 (13.56)
Physical activity (yes)	46 (13.9)	28 (60.87)	6 (13.04)
Cerebrovascular accident type (Ischemic)	266 (80.1)	177(66.54)	48 (18.05)

The multivariate L-M model (Table 2) showed that age (years), Sex, Employment status, History of myocardial infarction (employed), History of high cholesterol, Water pipe smoking, Past smoking were significant risk factors for death from BS (p¡0.1). In patients with other causes of death, History of blood pressure and Past smoking were significant risk factors ($p < 0.1$).

Table 2. Results of multivariate Lunn-McNeil modeling for risk factors of death from brain stroke and other causes

Characteristic	Brain Stroke		Other causes	
	Hazard ratio (90%CI)	p-value	Hazard ratio(90%CI)	p-value
Age category (years)				
<= 58	referent		referent	
59-68	1.50 (1.05-2.15)	0.000*	NA	
69-75	2.19 (1.58-3.03)	0.000*	NA	
76+	2.13 (1.52-3.01)	0.000*	NA	
Sex (female)	1.38 (1.08-1.76)	0.029*	NA	
Employment status (employed)	0.64(0.46-0.90)	0.034*	NA	
History of myocardial infarction	0.78 (0.50-1.21)	0.369	NA	
History of blood pressure	NA	NA	0.49(0.31-0.78)	0.021*
History of high cholesterol	0.62 (0.44-0.87)	0.021*	NA	
Water pipe smoking (yes)	2.78 (1.51-4.97)	0.005*	NA	
Past smoking (yes)	2.19 (1.70-2.81)	0.000*	1.64 (1.01-2.68)	0.095*
Cerebrovascular accident type (Ischemic)	NA	NA	1.75(0.84-3.66)	0.209

NA: not applicable; *$P < 0.1$; CI=Confidence interval

2.1 Comparison of mortality risk factors between BS and other causes of death

Significant differences were found between BS and other causes of death for the age category of 69-75 years, Employment status, Family history of stroke, a history of high blood pressure ($p < 0.1$) (Table 3).

Table 3. Comparison between brain stroke and other causes of death using the interaction Lunn-McNeil model

Characteristic	Hazard ratio	90% confidence interval	p-value
Age category (years)			
≤ 58	referent		
59-68	1.08	0.53-2.18	0.856
69-75	0.45	0.21-0.95	0.082*
76	0.62	0.28-1.34	0.309
Sex (female)	0.80	0.48-1.32	0.470
Education level (diploma)	0.99	0.54-1.81	0.988
Employment status (employed)	0.17	0.05-0.60	0.021*
Place of residence (city)	1.25	0.75-2.09	0.453
Family history of stroke	0.55	0.31-0.97	0.085*
Family history of heart attack	1.43	0.40-5.11	0.637
Heart disease	1.36	0.75-2.46	0.392
History of diabetes	1.20	0.62-2.34	0.639
History of blood pressure	2.59	1.55-4.34	0.002*
History of high cholesterol	0.75	0.39-1.43	0.466
Smoking (yes)	0.86	0.47-1.59	0.698
Past smoking (yes)	1.01	0.58-1.76	0.962
Passive smoking (yes)	1.38	0.66-2.85	0.465
Physical activity (yes)	1.08	0.49-2.38	0.867
Cerebrovascular accident type (Ischemic)	0.46	0.21-1.01	0.101

*$P < 0.1$; CI=Confidence interval

3 Discussion

The risk factors for mortality in patients with a BS diagnosis were compared between death from BS and death from other causes in the context of competing risks. We overcame the problem raised in the classical analysis (i.e., Cox regression) by utilizing the L-M approach. In classical analyses, it is usually assumed that competing risks are independent. Considering the competing risks as the censor and ignoring the risks incurred by other causes are the most crucial problems faced by the Cox model when analyzing data in the presence of competing risks. This leads to biased estimates [1]. The L-M model is a useful tool for analyzing data in the context of competing risks, in which the event occurs for several reasons. In this form of modeling, it is possible to compare the effect of each variable on competing causes of an event [4]. Therefore, in this study, the univariate and multivariate L-M approach was used, considering the risk of death from BS and death from other causes as competing risks.

Out of the 332 patients with a BS diagnosis, 227(68.4%) died from BS and 54 (16.3%) died from other causes. In line with our study, Hardy et al. assessed long-term (10-year) mortality after BS in Australia and reported a mortality rate of 79% [16]. The results of the univariate L-M analysis showed that an older age, Sex, Employment status for death from BS. These results align with those reported by studies conducted in Europe and the United States, according to which the highest mortality rate is found in the ninth decade of life [17, 18]. Furthermore, in a study conducted in Farshchian Hospital in Hamedan, patients under the age of 50 years had a lower risk of BS, and the risk of BS increased with age (19). Men were at a higher risk of death than women, which is consistent with the results of studies conducted in the United States [17], Europe [18], and Arab countries [20]. However, in some other studies, such as in the Copenhagen cohort study (2005), women were 1.49 times more likely to have died than men 10 years after a stroke [21]. This discrepancy may be due to physiological differences between men and women. In the study by Madsen et al., risk factors such as diabetes, metabolic syndrome, and migraine were found to increase the risk of stroke in women more than men, and hypertension was found to be associated with age and ethnicity [22].

In this study, History of myocardial infarction, History of blood pressure, History of high cholesterol, Water pipe smoking, Past smoking, Cerebrovascular accident type were significantly associated with death from BS, and History of blood pressure, Past smoking and Cerebrovascular accident type for death from Other. Hardy et al., in a study of 10-year survival after stroke, found that 79% of patients died within 10 years, and the leading cause of mortality was initial stroke and cardiovascular disease (27%) [23]. Another study showed that the overall mortality rate was 29%, and the mortality rate in those 70+ years of age was 57.1% [24]. ocp use is affected by age, blood pressure, smoking, and migraine [25]. A meta-analysis by Gillum et al. found that people on ocp had a higher risk of BS [26]. Stroke is divided into ischemic and hemorrhagic at the most basic level. In this study, approximately 80.1% of people had ischemic stroke, while 19.9% had hemorrhagic stroke. In line with our research, in a long-term attenuation study of stroke among people aged 18 to 50 years in the New Jersey region of the Netherlands in 2013 by Rutten et al., 606 of 959 (63%) patients had ischemic stroke [27]. In a 2016 study in Brazil, Goulart et al. found that the risk of death from hemorrhagic stroke was higher than that of ischemic stroke [28]. The multivariate L-M model showed that age (years), Sex, Employment status, History of myocardial infarction (employed), History of high cholesterol, Water pipe smoking, Past smoking were significant risk factors for death from BS ($p < 0.1$). In patients with other causes of death, History of blood pressure and Past smoking were significant risk factors ($p < 0.1$).

3.1 Comparison of mortality risk factors between BS and other causes of death

Significant differences were found between BS and other causes of death for the age category of 69-75 years, Employment status, Family history of stroke, a history of high blood pressure. According to the results of the multivariate L-M model, the significant risk factors for mortality due to BS were age (years), Sex, Employment status, History of myocardial infarction (employed), History of high cholesterol, Water pipe smoking, Past smoking. In patients with other causes of death, History of blood pressure and Past smoking were significant risk factors. Finally, according to the interaction L-M model, age category of 69-75 years, Employment status, Family history of stroke, a history of high blood pressure were significantly different between the 2 causes of death. The results of our study align with those of Mogensen et al., who studied the long-term (10 years) outcomes of stroke in 988 patients and concluded that stroke death was related to old age, sex, diabetes, a history of stroke, heart disease and non-vascular disease, and type of stroke (hemorrhagic) [29].

4 Conclusion

Differences were found between the 2 causes of death (BS and other causes) for some risk factors. Patients with BS were more likely to die from BS than from other causes. The hypothesis that some risk factors (demographic, clinical, and hereditary) may have different effects on the cause of death (BS and other causes) was confirmed based on the findings of this study. These findings should be considered for prevention, planning, health policymaking, and prescribing optimal and specific treatment in order to increase survival in patients with BS.

Ethical Considerations: The Ethics Committee of Tarbiat Modares University of Medical Sciences approved the protocol of this study (approval number: IR.MODARES.REC.1401.230).

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A brief study on series systems consisting of used components based on copulas

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Abstract

In this note, we consider series systems consisting of arbitrarily dependent used components. We study the lifetime of such systems using copulas family and obtain a formula for its survival function. In following, we provide stochastic ordering properties for the lifetimes of the series systems based on the mean function. Finally, to show the results we give a numerical example.

Keywords: Copula, Mean function, Stochastic order, Reliability.

1 Introduction

From a long time ago, many authors have studies on the coherent systems, especially series and parallel systems. It is well known that the reliability of a system depends on the structure of the system and obviously to the reliability of its components. The series system is a system with no redundancy which is used in various industries. Several researcher have tried to study on properties of this system according to various concepts and criteria. Most of the old results were gained on the condition of system components being independent. In practical perspectives we deal with to systems which have dependent components. Therefore, the analysis of dynamic reliability, particularly the study of the residual lifetime of systems, is the matter of interest in this case which

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is usually analyzed according to copulas theory. Copula function is a useful and efficient tools for describing the dependence structure between the components of a system. In recent decades, many researchers have focused their attention on this field, for instance you can see [4], [5], [7] and [8].

In some situations, it is possible for us to encounter systems consisting n-used components (see [1] and [2]). Therefore, the attractive issue in here is to make a comparison between the residual lifetimes of these systems from the theoretical viewpoint of dynamic reliability which is to be investigated in this paper for series systems.

The present article has been arranged as follows. Some required concepts and preliminaries to provide main results are presented in Section 2. In Section 3, at first the reliability function of the series systems consisting of used components was obtained according to survival copula supposing that the components were arbitrary. Then some properties of stochastic orderings for such systems was presented base on the mean function. Finally, we present an illustrative example and graphs to show the results.

2 Preliminaries

Before giving the main results,, we introduce the copula function and dependence concepts, for more details see [6]. Also, we need to present concepts of Schur-concave (Schur-convex) and mean function.

Definition 2.1. *A (n-variate) copula is a function $C : [0,1]^n \to [0,1]$ such that*

1. for any $u_i \in [0,1],\ C(1,\ldots,1,u_i,1,\ldots,1) = u_i$,
2. for any $u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_n \in [0,1], C(u_1,\ldots,u_{i-1},0,u_{i+1},\ldots,u_n) = 0$,
3. C is n-increasing.

For more details on an n-increasing function see [6] page 43. To present the next definitions we need to explain the majorization order. A vector $\mathbf{u} = (u_1,u_2,\ldots,u_n)$ is said to be majorized by another vector $\mathbf{v} = (v_1,v_2,\ldots,v_n)$ (written as $(u_1,u_2,\ldots,u_n) \leq_m (v_1,v_2,\ldots,v_n)$) if $\sum_{j=1}^n u_j = \sum_{j=1}^n v_j$ and $\sum_{j=1}^i u_{j:n} \geq \sum_{j=1}^i v_{j:n}$ for $i = 1,2,\ldots,n-1$, where $u_{j:n}$ ($v_{j:n}$) is the jth smallest element of $\mathbf{u}$ ($\mathbf{v}$), $j = 1,2,\ldots,n$.

Definition 2.2. Let $g : R^n \to R$ be a real-valued function, then g is Schur-concave (Schur-convex) if

$$g(u_1,u_2,\ldots,u_n) \leq (\geq) g(v_1,v_2,\ldots,v_n),$$

whenever $(u_1,u_2,\ldots,u_n) \geq_m (v_1,v_2,\ldots,v_n)$.

Definition 2.3. Let $g : R^n \to R$ be a real-valued function, then g is weakly Schur-concave (weakly Schur-convex) if

$$g(u_1,u_2,\ldots,u_n) \leq (\geq) g(\bar{u},\bar{u},\ldots,\bar{u}),$$

for all $(u_1,u_2,\ldots,u_n)$, where $\bar{u} = \frac{1}{n}\sum_{i=1}^n u_i$.

If a copula family is Schur-concave (Schur-convex), in fact it is weakly Schur-concave (weakly Schur-convex), too. In the following we give the concept of mean function that is useful to present the main results.

Definition 2.4. Let $g : R^n \to R$ be a real-valued function, then the mean function associated with g is any function $g : R^n \to R$ such that

$$g(u_1, u_2, \ldots, u_n) = g(z, z, \ldots, z),$$

for all $(u_1, u_2, \ldots, u_n)$, where $z = m_g(u_1, u_2, \ldots, u_n)$. For more details about mean function m_g see [3] and the references therein.

Definition 2.5. Let X and Y be two nonnegative random variables with survival functions $\bar{F}$ and $\bar{G}$, respectively. X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{\text{st}} Y$, if for all t, $\bar{F}(t) \leq \bar{G}(t)$.

3 Main results

Let $T_1, T_2, \ldots, T_n$ be the lifetimes of n components. Denote by $T_{1:n}, T_{2:n}, \ldots, T_{n:n}$ the ordered lifetimes of the components. It is known that the lifetime of series system is $T_{1:n}$. The joint survival function of a n dependent lifetimes of components denoted by

$$\begin{aligned} \bar{F}(t_1, t_2, \ldots, t_n) &= P(T_1 > t_1, T_2 > t_2, \ldots, T_n > t_n) \\ &= \hat{C}\Big(\bar{F}_1(t_1), \bar{F}_2(t_2), \ldots, \bar{F}_n(t_n)\Big), \end{aligned} \tag{1}$$

with marginal survival functions $\bar{F}_i(t) = P(T_i > t)$, $i = 1, 2, \ldots, n$.

The reliability analysis and stochastic ordering properties of systems consisting of used components of age $t > 0$ studied by [1], [2] and [9]. Let $(T_t)_i = (T_i - t | T_i > t)$, $i = 1, 2, \ldots, n$ be the residual lifetimes of used components, where $(T_t)_i$'s has the marginal survival function $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$, $i = 1, 2, \ldots, n$. Also, assume that T_i's have the survival copula $\hat{C}$. The survival function of the lifetime of the series system including used components denoted by $(\psi^T_{1:n}(x))_t$, is equal to

$$\begin{aligned} (\psi^T_{1:n}(x))_t &= P((T_t)_{1:n} > x) \\ &= \hat{C}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right) \\ &= \hat{C}(\kappa(x,t), \ldots, \kappa(x,t)) \end{aligned} \tag{2}$$

where $\kappa(x,t) = m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right)$, and $m_{\hat{C}}$ is the mean function of $\hat{C}$. Note that, the last equality in Equation (2) is obtained from Definition 2.4.

Now, we have the following results. The following theorems can be proved similarly to Theorems 3.4, and hence their proofs are omitted here.

Theorem 3.1. *Let $\mathbf{T} = (T_1, T_2, \ldots, T_n)$, denoted the arbitrary dependent lifetimes of the components of a series system. Let T_i's, $i = 1, 2, \ldots, n$, have the survival copula $\hat{C}$ with mean function $m_{\hat{C}}$. Then $(T_t)_{1:n} \leq_{st} (T_t)_{1:n-1}$ trues if and only if for all $x, t > 0$,*

$$m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right) \leq m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_{n-1}(x+t)}{\bar{F}_{n-1}(t)}\right),$$

where $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$ is the marginal survival function of $(T_t)_i$, $i = 1, 2, \ldots, n$.

Theorem 3.2. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ *and* $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ *be the arbitrary vectors of components lifetimes of two series systems. Also assume that* $\mathbf{T}$ *and* $\mathbf{Z}$ *have the same survival copula* $\hat{C}$ *with mean function* $m_{\hat{C}}$*. Then* $(T_t)_{1:n} \leq_{st} (Z_t)_{1:n}$ *if and only if for all* $x, t > 0$,

$$m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right) \leq m_{\hat{C}}\left(\frac{\bar{G}_1(x+t)}{\bar{G}_1(t)}, \ldots, \frac{\bar{G}_n(x+t)}{\bar{G}_n(t)}\right),$$

where $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$ *and* $\frac{\bar{G}_i(x+t)}{\bar{G}_i(t)}$ *are the marginal survival function of* $(T_t)_i$ *and* $(Z_t)_i$, $i = 1, 2, \ldots, n$, *respectively.*

Corollary 3.3. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ *be the arbitrary components lifetimes of a series system and* $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ *be the exchangeable components lifetimes of another series system. Also, assume that* $\mathbf{T}$ *and* $\mathbf{Z}$ *have the same survival copula* $\hat{C}$ *with mean function* $m_{\hat{C}}$*, then* $(T_t)_{1:n} \leq_{st} (Z_t)_{1:n}$ *trues if and only if for all* $x, t > 0$,

$$m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right) \leq \frac{\bar{G}(x+t)}{\bar{G}(t)},$$

where $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$ *and* $\frac{\bar{G}(x+t)}{\bar{G}(t)}$ *are the marginal survival function of* $(T_t)_i$ *and* $(Z_t)_i$, $i = 1, 2, \ldots, n$, *respectively.*

Theorem 3.4. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ $(\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n))$ *denote the arbitrary dependent lifetimes of the components of two series systems. Also, assume that* $\mathbf{T}$ *and* $\mathbf{Z}$ *have the survival copula* $\hat{C}_T$ *and* $\hat{C}_Z$*, respectively, such that* $\hat{C}_T \leq \hat{C}_Z$*. Then* $(T_t)_{1:n} \leq_{st} (Z_t)_{1:n}$*, if for all* $x, t > 0$,

$$m_{\hat{C}_T}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right) \leq m_{\hat{C}_Z}\left(\frac{\bar{G}_1(x+t)}{\bar{G}_1(t)}, \ldots, \frac{\bar{G}_n(x+t)}{\bar{G}_n(t)}\right), \tag{3}$$

where $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$ *and* $\frac{\bar{G}_i(x+t)}{\bar{G}_i(t)}$ *are the marginal survival function of* $(T_t)_i$ *and* $(Z_t)_i$, $i = 1, 2, \ldots, n$, *respectively.*

Proof. From (2) and after some simplifications we have for all $x, t > 0$,

$$\begin{aligned} (\psi^T_{1:n}(x))_t &= \hat{C}_T(\kappa_1(x,t), \ldots, \kappa_1(x,t)) \\ &\leq \hat{C}_T(\kappa_2(x,t), \ldots, \kappa_2(x,t)) \\ &\leq \hat{C}_Z(\kappa_2(x,t), \ldots, \kappa_2(x,t)) = (\psi^Z_{1:n}(x))_t, \end{aligned}$$

where $\kappa_1(x,t) = m_{\hat{C}_T}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right)$ and $\kappa_2(x,t) = m_{\hat{C}_Z}\left(\frac{\bar{G}_1(x+t)}{\bar{G}_1(t)}, \ldots, \frac{\bar{G}_n(x+t)}{\bar{G}_n(t)}\right)$. In the above expression, the first inequality follows from (3) and the second inequality obtain by $\hat{C}_T \leq \hat{C}_Z$. Therefore, the proof is complete. □

In the following we give a numerical example to apply the result of Theorem 3.4.

Example 3.5. Let $\mathbf{T} = (T_1, T_2)$ and $\mathbf{Z} = (Z_1, Z_2)$ be the vectors of component lifetimes of two series systems. Assume that $\mathbf{T}$ and $\mathbf{Z}$ have a Ali-Mikhail-Haq (AMH) bivariate survival copula $(\hat{C}_T)$ and Gumbel-Hougaard (GH) bivariate survival copula $(\hat{C}_Z)$, respectively, i.e.

$$\hat{C}_T(u,v) = \frac{uv}{1-\theta_1(1-u)(1-v)}, \qquad \theta_1 \in [-1,1] \tag{4}$$

$$\hat{C}_Z(u,v) = \exp\left\{-\left[(-\ln u)^{\theta_2} + (-\ln v)^{\theta_2}\right]^{\frac{1}{\theta_2}}\right\}, \quad \theta_2 \in [1,\infty). \tag{5}$$

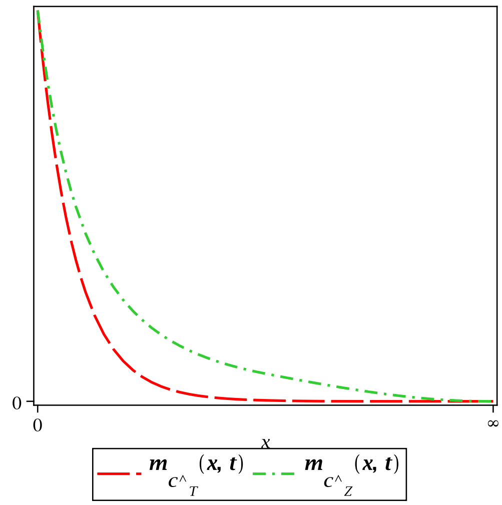

Figure 1: The curves of mean functions.

Since, the mean function of Archimedean copulas is equal to $\phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(u_i)\right)$ (See [5]), after some simplifications, the mean function of AMH survival copula for and GH survival copula are obtained, respectively, as follow:

$$m_{\hat{C}_T}(u,v) = \frac{1-\theta_1}{\sqrt{\left(\frac{1-\theta_1(1-u)}{u}\right)\left(\frac{1-\theta_1(1-v)}{v}\right)-\theta_1}}, \qquad \theta_1 \in [-1,1], \tag{6}$$

$$m_{\hat{C}_Z}(u,v) = \exp\left\{-\left[\frac{(-\ln u)^{\theta_2}+(-\ln v)^{\theta_2}}{2}\right]^{\frac{1}{\theta_2}}\right\}, \qquad \theta_2 \in [1,\infty). \tag{7}$$

Also, let T_i has exponential distribution with mean $\frac{1}{\lambda_i}$ and Z_i has Pareto distribution with marginal distribution function $F_i(x) = 1-(\frac{\beta_i}{x})^{\alpha_i}$, $i = 1,2$. From (6) and (7), we can obtain easily $m_{\hat{C}_T}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \frac{\bar{F}_2(x+t)}{\bar{F}_2(t)}\right)$ and $m_{\hat{C}_Z}\left(\frac{\bar{G}_1(x+t)}{\bar{G}_1(t)}, \frac{\bar{G}_2(x+t)}{\bar{G}_2(t)}\right)$.

On the other hand, from (2), the survival functions of two series system residual lifetimes are equal to

$$(\psi_{1:2}^{T}(x))_t = \frac{\bar{F}_1(x+t)\bar{F}_2(x+t)}{\bar{F}_1(t)\bar{F}_2(t)-\theta_1(\bar{F}_2(t)-\bar{F}_2(x+t))(\bar{F}_2(t)-\bar{F}_2(x+t))},$$

$$(\psi_{1:2}^{Z}(x))_t = \exp\left\{-\left[\left(-\ln\frac{\bar{G}_1(x+t)}{\bar{G}_1(t)}\right)^{\theta_2}+\left(-\ln\frac{\bar{G}_2(x+t)}{\bar{G}_2(t)}\right)^{\theta_2}\right]^{\frac{1}{\theta_2}}\right\}.$$

For $\theta_1 = 0.4$ and $\theta_2 = 3$ in (1) and (2), it can be shown that $\hat{C}_T(u,v) \le \hat{C}_Z(u,v)$. Furthermore, the graphs of mean functions $m_{\hat{C}_t,T}$ and $m_{\hat{C}_t,Z}$ are given Fig. 1,for $\lambda_1 = 1$, $\lambda_2 = 2$, $\alpha_1 = 3$, $\beta_1 = 1$, $\alpha_2 = 1$, $\beta_2 = 2$, $\theta_1 = 0.4$ and $\theta_2 = 3$ at the fixed point $t = 2$. Therefore, the conditions of Theorem 3.4 are satisfied. Also, we plot the graphs of the survival functions $(\psi_{1:2}^{T}(x))_t$ and $(\psi_{1:2}^{Z}(x))_t$ for the listed values of the parameters in Fig. 2. Hence, it is an application of Theorem 3.4.

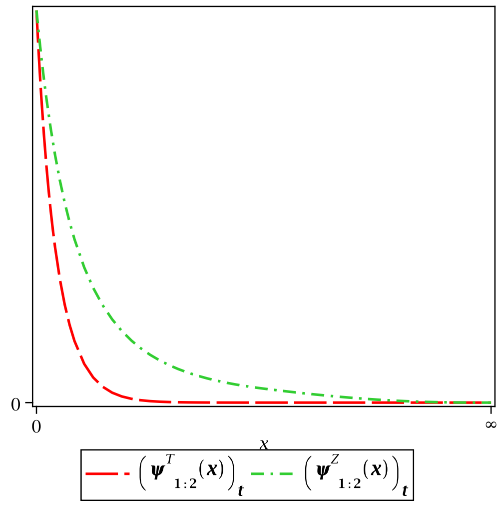

Figure 2: The curves of survival functions.

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The optimal age replacement strategy under epistemic uncertainty

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Abstract

Applying the theory of uncertainty is a good approach to study the reliability of a system when there is a little frequency of suitable data. Based on this theory, the policy of age replacement is considered assuming the system lifetime follows Weibull distribution. The unknown parameters are estimated according to evidence theory that affects the epistemic uncertainty. The Dempster-Shafer as well as Yager rules are applied to aggregate the judgements and mental estimates of two or more experts. After determining the unknown parameters, the optimal replacement time is derived using the long-run cost criterion. The results show that the Dempster-Shafer rule is more accurate than yager rule, but Yager's rule is more conservative than Dempster-Shafer's rule.

Keywords: Dempster-shafer theory; Maintenance policy; Uncertainty theory; Yager rule.

1 Introduction

When a system failure occurs, system access and reliability are reduced, leading to unexpected and catastrophic damage. Some systems have a set of core components that are expensive and we

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expect them to work without breakdowns for a long time. Due to their high price, these parts must be repaired when they fail. Therefore, by adopting the appropriate optimal maintenance policies and also identifying the sensitive units that need to be maintained, a high level of system reliability will be ensured and suitable models for optimizing system replacement and repair times can be provided.

In some reliability analyses, the probability distribution of unit lifetime data or the probability distribution of time to unit failure is not already known. In these cases, we have to estimate the probability distribution through a large number of samples. However, due to economic or technical issues, little or no data is available in some situations. As a result, the reliability of the system is unknown. Therefore, for such events for which it is not possible to measure frequency, the one way is to use the judgments and mental estimates of experts of the relevant field. Using this method, decision makers introduce more uncertainty approaches into the analysis due to insufficient knowledge of the credibility and knowledge of the requested experts (because humans tend to overestimate unlikely events). Therefore, the degree of belief usually has a wider range than the actual frequency. So, applying the theory of probability in such cases does not seem suitable, because of little frequency of available data in some real conditions.

Different theories have been proposed to solve this problem, such as fuzzy set theory by Zadeh [13], possibility theory by Zadeh [14] and Dubois and Prade [3], uncertainty theory by Liu [5, 6] and chance theory by Liu [8, 9].We use the uncertainty theory for lifetime and time data until units fail. Uncertainty theory was proposed by Liu [5] to deal with uncertainty caused by the degree of human belief in 2007. Regarding the use of experts' mental opinions, Siuta et al. [10], and Tsao [11] mention that mental opinions of the experts play a crucial role in uncertain environments and that experts are the only reliable and accessible source of information. There are various modern approaches to consider expert opinions such as fuzzy sets [13, 7], Dempster-Schaffer theory [2, 12] and theory of possibility [4]. Dempster-Shafer's theory (DST) of evidence was chosen as the appropriate mathematical framework to handle the presence of cognitive uncertainty. This theorem is a suitable theory to combine the opinions of two or more experts due to its very strong theoretical basis. Under the DST framework, expert judgments are transformed into a set of beliefs and, according to the combination rule of Dempster and Yager are assembled to determine the minimum and maximum reliability of the system. Assuming there are human uncertainties in the performance of systems, this paper discusses the age replacement policy introduced by Barlow and Hunter [1] based on the Dempster-Schaffer theorem. It also compares the results of the use of Dempster-Schaffer and Yager methods in this maintenance policy.

2 Preliminaries

The Dempster-Shafer Theory (DST) is a mathematical theory of evidence theory which is defined by three important functions: the Basic Probability Assignment (bpa) or belief mass, the Belief (Bel) and the Plausibility (Pl).

In evidence theory, for the sample space Θ with the power set P_Θ degrees of evidence are assigned to the subsets of $p_i \in P_\Theta$. Any subset p_i of P_Θ with non-zero evidence degree is called a focal element. The power set P_Θ is the set of all possible sub-sets of Θ including the empty set $\emptyset$. For example, if:

$$\Theta = \{a, b\}$$

Then

$$P_\Theta = \{\emptyset, \{a\}, \{b\}, \Theta\}$$

The *bpa*(A) represents the amount of knowledge associated with the subset A of P_Θ. Lets indicate by $m(A)$ the bpa related to every subset A. Each subset of the power set is assigned a mass value m between 0 and 1 according to the theory of evidence. The DST is the most suitable theory to combine the beliefs of two or more experts. Considering two experts and assuming the independence of their opinions, the DST is involved with joining two separate sets of mass functions on a frame of discernment Θ, such as m_1 and m_2. The combined mass function of m_1 and m_2 (the joint mass) in this situation would be represented as $m_{1,2}$ where:

$$m_{1,2}(A) = [m_1 \oplus m_2](A) = \begin{cases} 0, \quad for \quad A = \emptyset \\ \frac{\sum_{B\cap C=A} m_1(B).m_2(C)}{1-k}, \quad for \quad A \neq \emptyset \end{cases} \tag{1}$$

where $m_1(B)$ and $m_2(C)$ are the bpa expressed by the two sources of information 1 and 2 with relation to elements B and C respectively. Moreover, $k = \sum_{B\cap C=\emptyset} m_1(B).m_2(C)$ represents the conflicting evidence. To review the properties of $m(.)$, refer to [5].
If there is a lot of dispute among sources, some dubious findings might be obtained in DST. Therefore, the Yagers combination rule is alternatively used to aggregate the bpa associated to the set A, such that

$$[m_1 \oplus m_2](A) = \begin{cases} 0, \quad for \quad A = \emptyset \\ \sum_{B\cap C=A} m_1(B).m_2(C), \quad for \quad A \neq \Omega \\ \sum_{B\cap C=A} m_1(B).m_2(C) + k, \quad for \quad A = \Omega \end{cases} \tag{2}$$

3 Model description

Consider a system that has only one component and starts operating at t=0. Its lifetime cumulative distribution function (cdf) is $F(t)$. The system is replaced when it reaches age T, or at its failure, whichever occurs first. Assume that the emergency repair ER costs c_{ER}, and the preventive repair at age T costs c_{PM}. Each replacement completely renews the system and takes a negligible time. Denote by τ the unit's lifetime. Let $Z = \min(\tau, T)$. The inter-renewal period has mean

$$E(Z) = \int_0^T (1 - F(x))dx, \tag{3}$$

The mean cost per unit time is

$$\eta(T) = \frac{c_{ER}F(T) + c_{PM}(1 - F(T))}{\int_0^T (1 - F(x))dx} = c_{ER}\left(\frac{F(T) + c(1 - F(T))}{\int_0^T (1 - F(x))dx}\right), \tag{4}$$

where $c = \frac{c_{PM}}{c_{ER}} < 1$. For more details about the age replacement policy, see Barlow and Hunter [1]. Now suppose that the lifetime follows the two-parameter Weibull distribution with the probability density function (pdf):

$$f(t) = \frac{\beta}{\alpha}(\frac{t}{\alpha})^{\beta-1}\exp\{-(\frac{t}{\alpha})^\beta\}., \quad \beta, \alpha, t \geq 0 \tag{5}$$

and the cdf

$$F(t) = 1 - \exp\{-(\frac{t}{\alpha})^\beta\}. \tag{6}$$

From (4), we get

$$\eta(T) = c_{ER}\left(\frac{1+(c-1)\exp\{-(\frac{T}{\alpha})^{\beta}\}}{\int_0^T \exp\{-(\frac{x}{\alpha})^{\beta}\}dx}\right), \tag{7}$$

The main goal in this policy is to obtain the optimal replacemrnt time T, denoted by T^*, which minimizes the function (7). If the model parameters are known then the optimal T^* may be numerically obtained.
The cost function $\eta(T)$ of Equation (7) is clearly increasing relative to the parameter α.

When the model parameters are unknown, in reliability analysis, product life data (the lifetimes of the products operated successfully or time to failure and time to repair data) are used to estimate the unknown parameters and the reliability characteristics. These data points normally follow some probability distributions. They are required to identify the best-fit distribution and estimate the parameters of the distribution by some statistical methods like maximum likelihood estimation, least-square method and probability plotting. But, in many situations, these life data sets are not sufficient to use these methods of parameter estimation. In such cases, the model parameters can be alternatively determined by using the knowledge or experience of the experts which is called the expert judgment-based parameter estimation method.

4 Expert judgement based approach

As stated in the previous section, when no lifetime data are available to estimate the probability distribution parameters , we have to invite some domain experts to evaluate the belief degree that each event will happen. Perhaps some people think that the belief degree should be modeled by subjective probability or fuzzy set theory. However, it is usually inappropriate because both of them may lead to counterintuitive results in this case. In order to rationally deal with belief degrees, uncertainty theory was founded in 2007 and subsequently studied by many researchers. Nowadays, uncertainty theory has become a branch of axiomatic mathematics for modeling belief degrees.
The steps for applying the DST based approach to investigate the cognitive uncertainty of system reliability data are: First expert's judgments turn into a set of beliefs that can extract system-related credibility data according to each expert. Credibility data extracted from each expert's knowledge are then converted to bpa. Finally, the bpa values are aggregated according to the DST and Yager combination rules, and the system credibility limits can be obtained based on the combination of the opinions of all experts. Assume that the time to failure system follows a two-parameter Weibull distribution. If one of the parameters of the model is unknown, in order to use the expert judgment method to estimate the unknown parameter the following question is asked from the experts:
What is the average lifetime of the system a period of time?
Experts can not directly show the value of μ using a clear value because this value is strongly influenced by the specific operating conditions of the system under consideration. Instead, they are more realistically able to provide μ as an interval. Therefore, the eth ($e = 1, 2, 3, ..., E$) is asked to state the interval in which he/she believes the actual value of μ falls. For example, the data collected by expert e for the component is the interval $[\underline{\mu}, \overline{\mu}]_e$. In what follows, we assume β is known and α is unknown. Note that the mean of the distribution (5):

$$\mu = \alpha\Gamma(1 + \frac{1}{\beta}). \tag{8}$$

Therefore, the judged interval $[\underline{\mu}, \overline{\mu}]_e$ can be represented as follows

$$[\underline{\mu}, \overline{\mu}]_e = \left[\underline{\alpha}\Gamma(1+\frac{1}{\beta}), \overline{\alpha}\Gamma(1+\frac{1}{\beta})\right]_e \tag{9}$$

After collecting $[\underline{\mu}, \overline{\mu}]_e$, the interval value of the scale parameter$[\underline{\alpha}, \overline{\alpha}]_e$ can be calculated according to (9). Then, the lower and upper limit values of the reliability of the component in T_0, i.e. $[\underline{R}, \overline{R}]_e$, can be obtained using the following equation:

$$[\underline{R}(T_0), \overline{R}(T_0)]_e = \left[\exp\{-(\frac{T_0}{\underline{\alpha}})^\beta\}, \exp\{-(\frac{T_0}{\overline{\alpha}})^\beta\}\right]_e \tag{10}$$

Now we want to convert the reliability data calculated from the experience of experts to bpa. Assume that, the selected $FOD = \{W, F\}$ is discrete, where W and F respectively show the working and failure conditions of component . In addition, the resulting power set is $\{\emptyset, \{W\}, \{F\}, \{W, F\}\}$. Considering the DST framework, the maximum values of the interval $[\underline{R}, \overline{R}]_e$ can be explained as the minimum and maximum belief that the expert e relates it to the Working state event (i.e. W) for the component at the end of the working period T. so, these two bounds indicate Belief and Plausibility of this statement that the component is in the working condition W at the end of the working period T, the $\underline{R}_e = Bel(W)_e = m(W)_e$ and $\overline{R}_e = Pl(W)_e = 1 - m(F)_e$. Also $m(W, F)_e = 1 - m(W)_e - m(F)_e$. To illustrate the proposed procedure, an illustrative example is reported in the following. Lets assume a mission time T_0 equal to 200 time units. The parameter $\beta = 2$. The judgment intervals $[\underline{\mu}, \overline{\mu}]_e$ for the mean of distribution which are expressed by four experts (i.e. e = 1,2,3,4) are reported in Table 1. The corresponding values of the scale parameter $[\underline{\alpha}, \overline{\alpha}]_e$ are computed by using (9), and interval-valued reliabilities of the component are also shown in Table 1.

Table 1. Computation of reliability at time T_0 based on opinions of 4 experts

Expert	$\underline{\mu}$	$\overline{\mu}$	$\underline{\alpha}$	$\overline{\alpha}$	$\underline{R}(T_0)$	$\overline{R}(T_0)$
1	250	340	282.0948	383.6489	0.6049	0.7620
2	280	420	315.9464	473.9193	0.6698	0.8368
3	290	450	327.2300	507.7706	0.6883	0.8563
4	300	470	338.5138	530.3382	0.7053	0.8674

The valued of bpas are first computed based on the opinions of four expert, separately. Then, the bpas are aggregated by the Dempster and Yager rules using (1) and (2), respectively. The results are presented in Table2 and Table 3.

Table 2. The bpas for any expert

	$[\underline{R}(T_0), \overline{R}(T_0)]$		
e	$m_e(W)$	$m_e(F)$	$m_e(W, F)$
1	0.6049	0.2380	0.1571
2	0.6698	0.1632	0.1670
3	0.6883	0.1437	0.1680
4	0.7053	0.1326	0.1621

Table 3. The aggregated bpa based on opinions of 4 experts

Aggregated bpa	Dempster's rule	Yager's rule
$[m_1 \oplus m_2 \oplus m_3 \oplus m4](W)$	0.9753	0.4730
$[m_1 \oplus m_2 \oplus m_3 \oplus m4](F)$	0.0232	0.0113
$[m_1 \oplus m_2 \oplus m_3 \oplus m4](W, F)$	0.0015	0.5157

The lower bound of the components reliability at T_0 (i.e. $Bel(W)$) is the aggregated value of the $m(W)_e$ of the four experts, namely equal to $[m1 \oplus m2 \oplus m3 \oplus m4](W)$. The corresponding upper bound $Pl(W)$ is computed as $1-[m1 \oplus m2 \oplus m3 \oplus m4](F)$. Therefore, according to the aggregated data in Table 3, the interval $[\underline{R}(T_0), \overline{R}(T_0)]$ (credibility interval) of the working state event of the component at time $T_0 = 200$ based on Yager and DST rules are derived as

$$[\underline{R}, \overline{R}] = \begin{cases} [0.4730, 0.9887], & \text{Yager rule,} \\ [0.9753, 0.9768], & \text{DST rule.} \end{cases} \tag{11}$$

Hence, by solving the equations in (10), an interval-valued estimation for the scale parameter may be obtained as

$$[\underline{\alpha}, \overline{\alpha}] = \begin{cases} [231.1364, 1878.9620], & \text{Yager rule,} \\ [1264.5330, 1304.511], & \text{DST rule.} \end{cases} \tag{12}$$

5 Optimal maintenance problem

In this section we are going to optimize cost function (7) based on obtained credibility intervals for component under Yager and Dempester's rules. The optimal period of age replacement based on Yager and Dempster's rule are denoted by T_Y^* and T_D^*, respectively. The corresponding cost functions are also denoted by $\eta_Y(\cdot)$ and $\eta_D(\cdot)$, respectively. That is $\eta_Y(T_Y^*) = \min_T \eta_Y(T)$ and $\eta_D(T_D^*) = \min_T \eta_D(T)$.
To compare the performance of the Yager and Dempester rules, the following efficiency criterion is defined:

$$eff(\eta_Y, \eta_D) = \frac{\eta_D(T_D^*)}{\eta_Y(T_Y^*)} \tag{13}$$

According to the criterion (13), the Yager's rule is more efficient than Dempster's rule when $eff(\eta_Y, \eta_D) > 1$. To obtain the optimal replacemant time with a given shape parameter β and a given cost c for different values of T_0 in a large set $\{1, 2, 3, ..., 1000\}$ are initially used to determine a suitable judgement interval for the unknown parameter α. For different values of T_0, similar to the computational method used in Section 4, a judgement interval $[\underline{\alpha}_{T_0}, \overline{\alpha}_{T_0}]$ is determined for the scale parameter based on Yager and DST rules. We calculate the minimum cost and correspondingly the optimal value of replacemant time for each T_0. since the cost function $\eta_Y(T)$ is increasing in α, we use the minimum value of α to obtain the minimum value of the cost function. Therefore, the optimal value of replacement time for Yager and Dempester's rules may be derived. The results are presented in Table 4 for different values of c.

Table 4. Total expected Cost per unit time

	Yager		DST		
c	T_Y^*	$\eta_Y(T_Y^*)$	T_D^*	$\eta_D(T_D^*)$	eff
0.05	66.7986	151.0261	95.7358	105.3767	0.69773834
0.06	73.6315	164.7223	105.5287	114.9331	0.69773856
0.07	80.0343	177.4007	114.7053	123.5984	0.69671878
0.08	86.1085	188.5363	123.4109	131.5491	0.69773884
0.09	91.9246	199.3744	131.7465	138.9079	0.69671884
0.10	97.5341	208.9104	139.7860	145.7649	0.69773884
0.11	102.9760	218.1158	147.5854	152.8780	0.70090291
0.12	108.2810	226.7753	155.1884	158.2299	0.69773869
0.13	113.4734	234.9495	162.6303	163.9333	0.69773845
0.14	118.5736	242.6875	169.9398	169.3325	0.69773886
0.15	123.5984	250.0303	177.1413	174.4557	0.69773823

From Table 4, it is observed that:
1. As the value of c increases, the optimal replacement time for both Yager and Dempster rule's increases.
2. For all values of c, the optimal replacement time according to Dempster's rule T_D^* is greater than the optimal replacement time according to Yager's rule T_Y^*, but the minimum value of the cost function according to Dempster's rule $\eta_D(T_D^*)$ is slightly less than the minimum value of the cost function according to yager rule $\eta_Y(T_Y^*)$.
3. For all values of c, the value of efficiency is less than 1 and this means the Dempster's rule is more efficient than yager's rule.

6 Conclusions

An optimal age replacement strategy was studied under the epistemic uncertainty. For the situations in which the parameter of the distribution is unknown and little or no data are available to estimate the parameter, the Dempster-Shafer reasoning method was used to determine the model parameters based on the judgements and mental estimates of some experts in the relevant field. At first, the opinions of some experts were used to determine the unknown parameter about the average lifetime system. The knowledge of each expert was asked about the average lifetime of the system and the responses were converted to bpa values. Then, the obtained bpas for all experts were aggregated using the DST and Yager rules and they are used to specify the optimal maintenance policy. The results showed that DST rule is more accurate than yager rule, but yager rule works more conservatively than DST rule. The results of the paper may be extended to other maintenance model for a multicomponent system subject to more constraints.

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Usual stochastic order of α-mixtures with proportional odds model as a baseline

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Abstract

As a new flexible family of distributions, the α-mixture model includes many existing mixture models as special cases. This paper is an attempt to the usual stochastic order of this family when the underlying distribution follows from the proportional odds model. Sufficient conditions are provided for comparing two finite α-mixtures of survival functions with the baseline survival functions following the PO model in the sense of usual stochastic order when both the mixing proportions and the PO parameters of the first α-mixture majorize the mixing proportions and the PO parameters of the second one. Also, an upper bound for the reliability function of the α-mixture of survival function. Moreover, similar results are obtained for the α-mixture of cumulative distribution function. Finally, our theoretical findings are evaluated by some numerical examples.

Keywords: Mixture models, Usual stochastic order, Proportional odds model.

1 Introduction

In real life, one can hardly found homogeneous populations. In most areas, including the lifetime, the distribution of the lifetime populations is not homogeneous, and usually consist of a different number of homogeneous sub-populations. For example when components are mixed with two

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different product lines due to different work shifts, different raw materials, different manpower, etc. Obviously, due to the mentioned diversity in the production line, the lifetime distribution of components of one production line is different from another production line and when mixed, they will lead to heterogeneous populations. Ignoring the heterogeneity can lead to fundamental errors in reliability analysis. A new flexible family of distributions, called the α-mixture model proposed by [1], is usually an effective tool for modeling heterogeneity in populations, and includes many existing mixture models as special cases.

This short communications provides sufficient conditions for comparing two finite α-mixtures of survival (cumulative distribution) functions with the baseline distribution functions following the PO model in the sense of usual stochastic order when both the mixing proportions and the PO parameters of the first α-mixture majorize the mixing proportions and the PO parameters of the second one. The rest of the paper is organized as follows. In Section 2, the definition of the α-mixture model, proportional odds model and some basic definitions and lemmas are reviewed. Section 3, is devoted to the usual stochastic order of α-mixtures.

2 Preliminaries

Consider two random variables X and Y with SF's $\bar{F}$ and $\bar{G}$, respectively.

Definition 2.1. The random variable X is said to be smaller than Y in the usual stochastic order if $\bar{F}(x) \leq \bar{G}(x)$ for all x, or equivalently $E[\phi(X)] \leq E[\phi(Y)]$ for all increasing functions ϕ for which the expectations exist, and denoted by $X \leq_{st} Y$.

Definition 2.2. (Marshall et al. [2]). Let $a_{(1)} \leq \cdots \leq a_{(n)}$ and $b_{(1)} \leq \cdots \leq b_{(n)}$ denote the increasing arrangements of $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$, respectively.

(i) If $\sum_{j=1}^{i} a_{(j)} \leq \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \ldots, n-1$, and $\sum_{j=1}^{n} a_{(j)} = \sum_{j=1}^{n} b_{(j)}$, then $\boldsymbol{a}$ is said to majorize $\boldsymbol{b}$ and denoted by $\boldsymbol{a} \overset{m}{\succeq} \boldsymbol{b}$.

(ii) If $\sum_{j=1}^{i} a_{(j)} \leq \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \ldots, n$, then $\boldsymbol{a}$ is said to weakly supermajorize $\boldsymbol{b}$, and denoted by $\boldsymbol{a} \overset{w}{\succeq} \boldsymbol{b}$.

(iii) If $\sum_{j=i}^{n} a_{(j)} \geq \sum_{j=i}^{n} b_{(j)}$ for $i = 1, \ldots, n$, then $\boldsymbol{a}$ is said to weakly submajorize $\boldsymbol{b}$, denoted by $\boldsymbol{a} \succeq_w \boldsymbol{b}$.

Definition 2.3. (Marshall et al. [2]). Consider a real-valued function ϕ defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$. If $\boldsymbol{a} \overset{m}{\succeq} \boldsymbol{b}$ implies $\phi(\boldsymbol{a}) \geq (\leq) \phi(\boldsymbol{b})$ for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{A}$, then ϕ is Schur-convex (Schur-concave) on $\mathbb{A}$.

Characterizations of Schur-convex (Schur-concave) functions are provided in the next lemma.

Lemma 2.4. *(Marshall et al. [2]). Suppose that $\phi : I^n \to \mathbb{R}$ be a real-valued, continuously differentiable function, where $I \subseteq \mathbb{R}$ is an open interval. Then, ϕ is Schur-convex (Schur-concave) on I^n if and only if*

(i) ϕ is symmetric on I^n, and

(ii) for all $i \neq j$ and all $\boldsymbol{x} \in I^n$,

$$(a_i - a_j)\left(\frac{\partial \phi}{\partial a_i}(\boldsymbol{a}) - \frac{\partial \phi}{\partial a_j}(\boldsymbol{a})\right) \geq (\leq) 0,$$

where $\frac{\partial \phi}{\partial a_i}$ is the partial derivative of ϕ with respect to its i-th argument.

Lemma 2.5. *(Marshall et al. [2]). Let ϕ be a real-valued function defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$. Then,*

(i) ϕ is increasing and Schur-convex on $\mathbb{A}$ if and only if $\boldsymbol{a} \succeq_w \boldsymbol{b}$ implies $\phi(\boldsymbol{a}) \geq \phi(\boldsymbol{b})$;

(ii) ϕ is decreasing and Schur-convex on $\mathbb{A}$ if and only if $\boldsymbol{a} \stackrel{w}{\succeq} \boldsymbol{b}$ implies $\phi(\boldsymbol{a}) \geq \phi(\boldsymbol{b})$.

Before giving the main results of the paper, we need the following notation:

$$\mathcal{S}_n = \{(\boldsymbol{a}, \boldsymbol{b}) : a_i, b_i \leq 0 \quad (a_i - a_j)(b_i - b_j) \leq 0, \quad i, j = 1, \ldots, n\}.$$

2.1 Proportional odds model

The proportional odds (PO) model, that proposed by Bennett [2], is a very important semi-parametric model in reliability theory and survival analysis. A random variable $T|\lambda$ is said to follow a PO model if its SF is expressed as

$$\bar{F}(t|\lambda) = \frac{\lambda \bar{F}(t)}{1 - \bar{\lambda}\bar{F}(t)},$$

where $\lambda > 0$ is a constant, and $\bar{F}(t)$ is the baseline SF. In this case the hazard rate function of T is

$$r(t|\lambda) = \frac{r(t)}{1 - \bar{\lambda}\bar{F}(t)},$$

where $r(t)$ is the baseline hazard rate. Note that the odds function of random variable $T|\lambda$, denoted by $\tau(t|\lambda)$, is $\tau(t|\lambda) = \lambda\tau(t)$, where $\tau(t)$ is the baseline odds. Similarly the CDF of $T|\lambda$ is

$$F(t|\lambda) = \frac{F(t)}{1 - \bar{\lambda}\bar{F}(t)},$$

where $F(t)$ is the baseline CDF.

2.2 Finite α-Mixture

Following [1] the finite α-mixture of SF's for n homogeneous subpopulations with SF's $\bar{F}_i$, $i = 1, 2, .., n$, is

$$\bar{F}_\alpha(t) = \begin{cases} \left[\sum_{i=1}^n p_i \bar{F}_i^\alpha(t)\right]^{1/\alpha}, & \alpha \neq 0, \\ \bar{F}_{gm}(t) = \prod_{i=1}^n \bar{F}_i^{p_i}(t), & \alpha = 0, \end{cases} \tag{1}$$

where $\bar{F}_{gm}(t) = \lim_{\alpha \to 0} \bar{F}_\alpha(t)$ is the geometric mean of SF's $\bar{F}_i$, and $p_i \geq 0$, $i = 1, \ldots, n$, is the mixing proportion such that $\sum_{i=1}^n p_i = 1$.

If we denote by $r_\alpha(t)$ and $r_i(t)$ the hazard rate of the finite α-mixture of SF's and hazard rate of the i-th subpopulation, respectively, then

$$r_\alpha(t) = \sum_{i=1}^n r_i(t) p_i(t), \tag{2}$$

where $p_i(t) = \frac{p_i \bar{F}_i^\alpha(t)}{\sum_{i=1}^n p_i \bar{F}_i^\alpha(t)}$. In particular, for the finite α-mixture of SF's (1), the hazard rate of the geometric mixture, denoted by $r_{gm}(t)$ can be expressed as:

$$r_{gm}(t) = \sum_{i=1}^n p_i r_i(t).$$

For some reliability interpretations of the finite α-mixture of SF's for different values of α, interested readers may refer to [4] and [5].

Similarly, the finite α-mixture of CDF's for n homogeneous subpopulations with CDF's F_i, $i = 1, 2, .., n$, can be defined as

$$F_\alpha(t) = \begin{cases} \left[\sum_{i=1}^n p_i {F_i}^\alpha(t)\right]^{1/\alpha}, & \alpha \neq 0, \\ F_{gm}(t) = \prod_{i=1}^n F_i^{p_i}(t), & \alpha = 0, \end{cases} \tag{3}$$

where $F_{gm}(t) = \lim_{\alpha \to 0} F_\alpha(t)$.

3 Main results

This section compares the reliability functions of two finite α-mixtures of SF's (CDF's) with the baseline SF's (CDF's) following the PO model in the sense of usual stochastic order when both the mixing proportions and the parameters of the first α-mixture majorize the mixing proportions and the parameters of the second one. Since the case $\alpha \to 0$, the geometric mean of SF's (CDF's), provides an entirely different model, the corresponding results are presented separately.

Theorem 3.1. *Suppose that $M_\alpha(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$ and $M_\alpha^\star(\boldsymbol{q}, \boldsymbol{\gamma}, PO)$ are the lifetime random variables of two n-component finite α-mixtures of SF's with the respective proportional odds parameters $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$, and the respective mixing proportions $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ with SF's $\bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t) = \left[\sum_{i=1}^n p_i \left(\frac{\lambda_i \bar{F}(t)}{1-\bar{\lambda}_i \bar{F}(t)}\right)^\alpha\right]^{1/\alpha}$ and $\bar{F}_{M_\alpha^\star(\boldsymbol{q},\boldsymbol{\gamma},PO)}(t) = \left[\sum_{i=1}^n q_i \left(\frac{\gamma_i \bar{F}(t)}{1-\bar{\gamma}_i \bar{F}(t)}\right)^\alpha\right]^{1/\alpha}$, respectively. If $\boldsymbol{p} \succeq_w \boldsymbol{q}$ ($\boldsymbol{p} \overset{w}{\succeq} \boldsymbol{q}$) and $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\gamma}$, then for $(\boldsymbol{p}, \boldsymbol{\lambda}) \in \mathcal{S}_n$ and $(\boldsymbol{q}, \boldsymbol{\gamma}) \in \mathcal{S}_n$ we have: $M_\alpha(\boldsymbol{p}, \boldsymbol{\lambda}, PO) \leq_{st} M_\alpha^\star(\boldsymbol{q}, \boldsymbol{\gamma}, PO)$ for $\alpha < 0$ ($\alpha \in (0, 1]$).*

Proof. To prove the theorem, it is enough to show that $\bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)$ is decreasing (increasing) and Schur-concave with respect to $\boldsymbol{\lambda}$. The partial derivative of $\bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)$ with respect to λ_i is

$$\frac{\partial \bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)}{\partial \lambda_i} = p_i \frac{\partial \bar{F}(t|\lambda_i)}{\partial \lambda_i} \bar{F}^{\alpha-1}(t|\lambda_i) \left[\sum_{i=1}^n p_i \bar{F}^\alpha(t|\lambda_i)\right]^{\frac{1}{\alpha}-1} \geq 0,$$

where $\bar{F}(t|\lambda_i) = \frac{\lambda_i \bar{F}(t)}{1-\bar{\lambda}_i \bar{F}(t)}$ is increasing in λ. Thus, $\bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)$ is increasing in λ_i. Now, for any $u \neq s$, $u, s \in \{1, \ldots, n\}$

$$\begin{aligned} &(\lambda_u - \lambda_s)\left[\frac{\partial \bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)}{\partial \lambda_u} - \frac{\partial \bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)}{\partial \lambda_s}\right] \\ &= (\lambda_u - \lambda_s)\left[p_u \frac{\partial \bar{F}(t|\lambda_u)}{\partial \lambda_u} \bar{F}^{\alpha-1}(t|\lambda_u) - p_s \frac{\partial \bar{F}(t|\lambda_s)}{\partial \lambda_s} \bar{F}^{\alpha-1}(t|\lambda_s)\right]. \end{aligned}$$

Now, for $\lambda_u \leq (\geq)\lambda_s$, $\bar{F}^{\alpha-1}(t|\lambda_u) \geq (\leq)\bar{F}^{\alpha-1}(t|\lambda_s)$. Since $\bar{F}(t|\lambda)$ is increasing and concave, we get: $\frac{\partial \bar{F}(t|\lambda_u)}{\partial \lambda_u} \geq (\leq)\frac{\partial \bar{F}(t|\lambda_s)}{\partial \lambda_s}$. On the other hand, from $(\boldsymbol{p}, \boldsymbol{\lambda}) \in \mathcal{S}_n$, we have $p_u \geq (\leq)p_s$. Consequently,

$$(\lambda_u - \lambda_s)\left[\frac{\partial \bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)}{\partial \lambda_u} - \frac{\partial \bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)}{\partial \lambda_s}\right] \leq 0.$$

Thus, using Lemma 2.4, $\bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)$ is Schur-concave with respect to $\boldsymbol{\lambda}$. Hence, by Lemma 2.5 if $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\gamma}$, then

$$\left[\sum_{i=1}^{n} p_i \bar{F}^{\alpha}(t|\lambda_i)\right]^{1/\alpha} \leq \left[\sum_{i=1}^{n} p_i \bar{F}^{\alpha}(t|\gamma_i)\right]^{1/\alpha}. \tag{4}$$

That means $M_\alpha(\boldsymbol{p}, \boldsymbol{\lambda}, PO) \leq_{st} M_\alpha(\boldsymbol{p}, \boldsymbol{\gamma}, PO)$ for $\alpha < 0$ ($\alpha \in (0, 1]$). Now, we have must to show that $M_\alpha(\boldsymbol{p}, \boldsymbol{\gamma}, PO) \leq_{st} M_\alpha(\boldsymbol{q}, \boldsymbol{\gamma}, PO)$. Thus, we have

$$\frac{\partial \bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\gamma},PO)}(t)}{\partial p_i} = \frac{1}{\alpha}\bar{F}^{\alpha}(t|\gamma_i)\left[\sum_{i=1}^{n} p_i \bar{F}^{\alpha}(t|\gamma_i)\right]^{\frac{1}{\alpha}-1} \leq (\geq)0,$$

for $\alpha < 0$ ($\alpha \in (0, 1]$). On the other hand,

$$\begin{aligned}(p_u - p_s)&\left[\frac{\partial \bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\gamma},PO)}(t)}{\partial p_u} - \frac{\partial \bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\gamma},PO)}(t)}{\partial p_s}\right] \\ &\overset{sign}{=} (p_u - p_s)\left[\frac{1}{\alpha}\big(\bar{F}^{\alpha}(t|\gamma_u) - \bar{F}^{\alpha}(t|\gamma_s)\big)\right] \leq 0.\end{aligned}$$

Thus, $\bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\gamma},PO)}(t)$ is decreasing (increasing) and Schur-concave with respect to $\boldsymbol{p}$ for $\alpha < 0$ ($\alpha \in (0, 1]$). Hence, if $\boldsymbol{p} \succeq_w \boldsymbol{q}$ ($\boldsymbol{p} \overset{w}{\succeq} \boldsymbol{q}$), then Lemma 2.5 yields

$$\left[\sum_{i=1}^{n} p_i \bar{F}^{\alpha}(t|\gamma_i)\right]^{1/\alpha} \leq \left[\sum_{i=1}^{n} q_i \bar{F}^{\alpha}(t|\gamma_i)\right]^{1/\alpha}, \tag{5}$$

for $\alpha < 0$ ($\alpha \in (0, 1]$). By combining (6) and (7), we get: $M_\alpha(\boldsymbol{p}, \boldsymbol{\lambda}, PO) \leq_{st} M^{\star}_\alpha(\boldsymbol{q}, \boldsymbol{\gamma}, PO)$, completing the proof of the theorem. □

Theorem 3.1 gives the following upper bound for the SF of the random variable $M_\alpha(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$.

Corollary 3.2. Set $(\gamma_1, \ldots, \gamma_n) = (\bar{\lambda}, \ldots, \bar{\lambda})$, where $\bar{\lambda} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i$, and $(q_1, \ldots, q_n) = (\bar{p}, \ldots, \bar{p})$, where $\bar{p} = \frac{1}{n}\sum_{i=1}^{n} p_i$. It is easy to see that $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\gamma}$ and $\boldsymbol{p} \succeq_w \boldsymbol{q}$ ($\boldsymbol{p} \overset{w}{\succeq} \boldsymbol{q}$). Thus, an upper bound for the SF of $M_\alpha(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$ for $\alpha \in (-\infty, 0) \cup (0, 1)$ is as follows.

$$\bar{F}_{M_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t) \leq (n\bar{p})^{\frac{1}{\alpha}} \frac{\bar{\lambda}\bar{F}(t)}{1 - \bar{\bar{\lambda}}\bar{F}(t)},$$

where $\bar{\bar{\lambda}} = 1 - \bar{\lambda}$.

The following theorem compares two geometric mixtures of SF's with PO model as baselines.

Theorem 3.3. *Suppose that $G(\boldsymbol{p},\boldsymbol{\lambda},PO)$ and $G^\star(\boldsymbol{q},\boldsymbol{\gamma},PO)$ are the lifetime random variables of two n-component finite geometric mixtures of SF's with the respective proportional odds parameters $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$ and $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_n)$, and the respective mixing proportions $\boldsymbol{p}=(p_1,\ldots,p_n)$ and $\boldsymbol{q}=(q_1,\ldots,q_n)$ with SF's $\bar F_{G(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)=\prod_{i=1}^n\left(\frac{\lambda_i\bar F(t)}{1-\bar\lambda_i\bar F(t)}\right)^{p_i}$ and $\bar F_{G^\star(\boldsymbol{q},\boldsymbol{\gamma},PO)}(t)=\prod_{i=1}^n\left(\frac{\gamma_i\bar F(t)}{1-\bar\gamma_i\bar F(t)}\right)^{q_i}$, respectively. If $\boldsymbol{p}\succeq_w\boldsymbol{q}$ and $\boldsymbol{\lambda}\overset{w}{\succeq}\boldsymbol{\gamma}$, then for $(\boldsymbol{p},\boldsymbol{\lambda})\in\mathcal{S}_n$ and $(\boldsymbol{q},\boldsymbol{\gamma})\in\mathcal{S}_n$ we have: $G(\boldsymbol{p},\boldsymbol{\lambda},PO)\le_{st}G^\star(\boldsymbol{q},\boldsymbol{\gamma},PO)$.*

Proof. To prove the theorem, it is enough to show that $\bar F_{G(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)$ is decreasing and Schur-concave with respect to $\boldsymbol{\lambda}$. Similar to proof of Theorem 3.1, the partial derivative of $\bar F_{G(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)$ with respect to λ_i is

$$\frac{\partial\bar F_{G(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)}{\partial\lambda_i}=p_i\frac{\frac{\partial\bar F(t|\lambda_i)}{\partial\lambda_i}}{\bar F(t|\lambda_i)}\bar F_{G(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)\ge 0,$$

because $\bar F(t|\lambda_i)$ is increasing in λ. For any $u\neq s$, $u,s\in\{1,\ldots,n\}$

$$\begin{aligned}&(\lambda_u-\lambda_s)\left[\frac{\partial\bar F_{G(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)}{\partial\lambda_u}-\frac{\partial\bar F_{G(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)}{\partial\lambda_s}\right]\\ &\overset{sign}{=}(\lambda_u-\lambda_s)\left[p_u\frac{\frac{\partial\bar F(t|\lambda_u)}{\partial\lambda_u}}{\bar F(t|\lambda_u)}-p_s\frac{\frac{\partial\bar F(t|\lambda_s)}{\partial\lambda_s}}{\bar F(t|\lambda_s)}\right]\le 0.\end{aligned}$$

Hence, by Lemma 2.4, $\bar F_{G(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)$ is Schur-concave with respect to $\boldsymbol{\lambda}$, and if $\boldsymbol{\lambda}\overset{w}{\succeq}\boldsymbol{\gamma}$, then Lemma 2.5 yields

$$\prod_{i=1}^n\bar F^{p_i}(t|\lambda_i)\le\prod_{i=1}^n\bar F^{p_i}(t|\gamma_i). \tag{6}$$

On the other hand,

$$\frac{\partial\bar F_{G(\boldsymbol{p},\boldsymbol{\gamma},PO)}(t)}{\partial p_i}=\log\bar F(t|\gamma_i)\bar F_{G(\boldsymbol{p},\boldsymbol{\gamma},PO)}(t)\le 0,$$

and

$$(p_u-p_s)\left[\frac{\partial\bar F_{G(\boldsymbol{p},\boldsymbol{\gamma},PO)}(t)}{\partial p_u}-\frac{\partial\bar F_{G(\boldsymbol{p},\boldsymbol{\gamma},PO)}(t)}{\partial p_s}\right]\le 0.$$

Thus, $\bar F_{G(\boldsymbol{p},\boldsymbol{\gamma},PO)}(t)$ is decreasing and Schur-concave with respect to $\boldsymbol{p}$. Hence, from Lemma 2.5, if $\boldsymbol{p}\succeq_w\boldsymbol{q}$, then

$$\prod_{i=1}^n\bar F^{p_i}(t|\gamma_i)\le\prod_{i=1}^n\bar F^{q_i}(t|\gamma_i). \tag{7}$$

Consequently, from (6) and (7), one can obtain: $G(\boldsymbol{p},\boldsymbol{\lambda},PO)\le_{st}G^\star(\boldsymbol{q},\boldsymbol{\gamma},PO)$. ☐

Corollary 3.4. Set $(\gamma_1,\ldots,\gamma_n)=(\bar\lambda,\ldots,\bar\lambda)$, where $\bar\lambda=\frac{1}{n}\sum_{i=1}^n\lambda_i$, and $(q_1,\ldots,q_n)=(\bar p,\ldots,\bar p)$, where $\bar p=\frac{1}{n}\sum_{i=1}^n p_i$. It is easy to see that $\boldsymbol{\lambda}\overset{w}{\succeq}\boldsymbol{\gamma}$ and $\boldsymbol{p}\succeq_w\boldsymbol{q}$. Thus, an upper bound for the SF of $G(\boldsymbol{p},\boldsymbol{\lambda},PO)$ is as follows.

$$\bar F_{G(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)\le\left(\frac{\bar\lambda\bar F(t)}{1-\bar{\bar\lambda}\bar F(t)}\right)^{n\bar p},$$

where $\bar{\bar\lambda}=1-\bar\lambda$.

In the following, similar results for the finite α-mixture of CDF's, without proofs, are provided.

Theorem 3.5. *Let $W_\alpha(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$ and $W^\star_\alpha(\boldsymbol{q}, \boldsymbol{\gamma}, PO)$ be the lifetime random variables of two n-component finite α-mixtures of CDF's with the respective proportional odds parameters $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$, and the respective mixing proportions $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ with CDF's $F_{W_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t) = \left[\sum_{i=1}^n p_i \left(\frac{F(t)}{1-\bar{\lambda}_i \bar{F}(t)}\right)^\alpha\right]^{1/\alpha}$ and $F_{W^\star_\alpha(\boldsymbol{q},\boldsymbol{\gamma},PO)}(t) = \left[\sum_{i=1}^n q_i \left(\frac{F(t)}{1-\bar{\gamma}_i \bar{F}(t)}\right)^\alpha\right]^{1/\alpha}$ respectively. If $\boldsymbol{p} \succeq_w \boldsymbol{q}$ and $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\gamma}$, then for $(\boldsymbol{p}, \boldsymbol{\lambda}) \in \mathcal{S}_n$ and $(\boldsymbol{q}, \boldsymbol{\gamma}) \in \mathcal{S}_n$ we have: $W_\alpha(\boldsymbol{p}, \boldsymbol{\lambda}, PO) \leq_{st} W^\star_\alpha(\boldsymbol{q}, \boldsymbol{\gamma}, PO)$ for $\alpha \geq 1$.*

Corollary 3.6. Set $(\gamma_1, \ldots, \gamma_n) = (\bar{\lambda}, \ldots, \bar{\lambda})$, where $\bar{\lambda} = \frac{1}{n}\sum_{i=1}^n \lambda_i$, and $(q_1, \ldots, q_n) = (\bar{p}, \ldots, \bar{p})$, where $\bar{p} = \frac{1}{n}\sum_{i=1}^n p_i$. It is easy to see that $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\gamma}$ and $\boldsymbol{p} \succeq_w \boldsymbol{q}$. Thus, a lower bound for the CDF of $W_\alpha(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$ for $\alpha \geq 1$ is as follows.

$$(n\bar{p})^{\frac{1}{\alpha}} \frac{F(t)}{1 - \bar{\bar{\lambda}}\bar{F}(t)} \leq F_{W_\alpha(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t),$$

where $\bar{\bar{\lambda}} = 1 - \bar{\lambda}$.

Theorem 3.7. *Let $G_c(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$ and $G^\star_c(\boldsymbol{q}, \boldsymbol{\gamma}, PO)$ be the lifetime random variables of two n-component finite geometric mixtures of CDF's with the respective proportional odds parameters $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$, and the respective mixing proportions $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ with CDF's $F_{G_c(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t) = \prod_{i=1}^n \left(\frac{F(t)}{1-\bar{\lambda}_i \bar{F}(t)}\right)^{p_i}$ and $F_{G^\star_c(\boldsymbol{q},\boldsymbol{\gamma},PO)}(t) = \prod_{i=1}^n \left(\frac{F(t)}{1-\bar{\gamma}_i \bar{F}(t)}\right)^{q_i}$, respectively. If $\boldsymbol{p} \overset{w}{\succeq} \boldsymbol{q}$ and $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\gamma}$, then for $(\boldsymbol{p}, \boldsymbol{\lambda}) \in \mathcal{S}_n$ and $(\boldsymbol{q}, \boldsymbol{\gamma}) \in \mathcal{S}_n$ we have: $G_c(\boldsymbol{p}, \boldsymbol{\lambda}, PO) \leq_{st} G^\star_c(\boldsymbol{q}, \boldsymbol{\gamma}, PO)$.*

Corollary 3.8. Set $(\gamma_1, \ldots, \gamma_n) = (\bar{\lambda}, \ldots, \bar{\lambda})$, where $\bar{\lambda} = \frac{1}{n}\sum_{i=1}^n \lambda_i$, and $(q_1, \ldots, q_n) = (\bar{p}, \ldots, \bar{p})$, where $\bar{p} = \frac{1}{n}\sum_{i=1}^n p_i$. It is easy to see that $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\gamma}$ and $\boldsymbol{p} \overset{w}{\succeq} \boldsymbol{q}$. Thus, an upper bound for the CDF of $G_c(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$ is as follows.

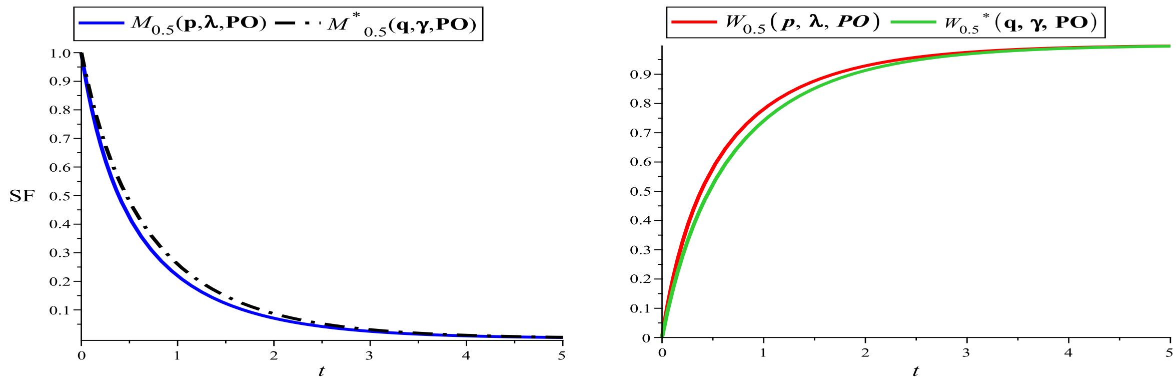
$$F_{G_c(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t) \leq \left(\frac{F(t)}{1 - \bar{\bar{\lambda}}\bar{F}(t)}\right)^{n\bar{p}},$$

where $\bar{\bar{\lambda}} = 1 - \bar{\lambda}$.

Let us consider some examples for our theoretical findings.

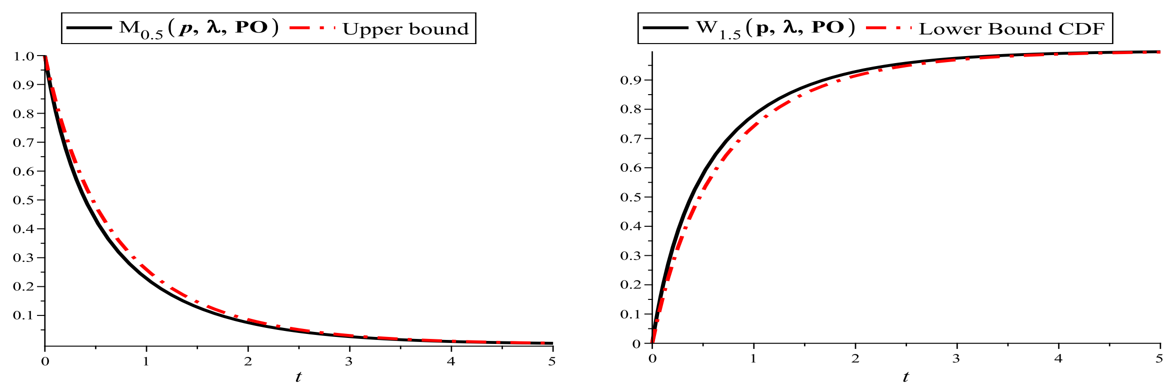
Example 3.9. (a) Suppose that the baseline distribution is standard Exponential distribution with SF $\bar{F}(t) = e^{-t}$, $t \geq 0$. Set $\boldsymbol{p} = (p_1, p_2, p_3) = (0.7, 0.2, 0.1)$, $\boldsymbol{q} = (q_1, q_2, q_3) = (0.6, 0.2, 0.2)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (0.4, 0.7, 0.8)$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = (0.5, 0.7, 0.9)$ and $\alpha = 0.5$. Clearly, $\boldsymbol{p} \overset{w}{\succeq} \boldsymbol{q}$, $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\gamma}$, $(\boldsymbol{p}, \boldsymbol{\lambda}) \in \mathcal{S}_3$ and $(\boldsymbol{q}, \boldsymbol{\gamma}) \in \mathcal{S}_3$. Thus, all condition of Theorem 3.1 are satisfied. Figure 1 (a) depicts the SF's of $M_{0.5}(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$ and $M^\star_{0.5}(\boldsymbol{q}, \boldsymbol{\gamma}, PO)$, and indicates that the usual stochastic order is hold. Also, Figure 2 (a) Shows the SF's of $M_{0.5}(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$ and its corresponding upper bound given in Corollary 3.2.

(b) Let $W_\alpha(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$ and $W^\star_\alpha(\boldsymbol{q}, \boldsymbol{\gamma}, PO)$ are two finite α-mixtures of CDF's with the specifications given as part (a) and $\alpha = 1.5$. Figure 1 (b) plots the CDF's of $W_{1.5}(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$ and $W^\star_{1.5}(\boldsymbol{q}, \boldsymbol{\gamma}, PO)$, and coincides with the result of Theorem 3.5. Further, Figure 2 (b) plots the CDF's of $W_{1.5}(\boldsymbol{p}, \boldsymbol{\lambda}, PO)$ and its corresponding lower bound given in Corollary 3.6.



(a) (b)

Figure 1: (a): $\bar{F}_{M_{0.5}(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)$ (solid) and $\bar{F}_{M^\star_{0.5}(\boldsymbol{q},\boldsymbol{\gamma},PO)}(t)$ (dash dot); and (b): $F_{W_{1.5}(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)$ (solid) and $F_{W^\star_{1.5}(\boldsymbol{q},\boldsymbol{\gamma},PO)}(t)$ (dash dot) in Example 3.9.



(a) (b)

Figure 2: (a): $\bar{F}_{M_{0.5}(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)$ (solid) and the corresponding upper bound (dash dot); and (b): $F_{W_{1.5}(\boldsymbol{p},\boldsymbol{\lambda},PO)}(t)$ (solid) and the corresponding lower bound (dash dot) in Example 3.9.

4 Conclusions

In this paper, we have considered the PO model as a baseline distribution in the finite α-mixture model, and have provided sufficient conditions, using the concept of majorization, to compare two finite α-mixtures in the sense of usual stochastic order. Recently, Shojaee et al. [6] have compared two generalized finite α-mixtures of SF's when the baseline SF is decreasing (increasing) and convex (concave) in the parameter for $\alpha_i \geq 0$ ($\alpha_i \leq 0$). The finite α-mixture is the special case of the generalized finite α-mixture with $\alpha_i = \alpha$, however, the results of the paper have obtained under slightly different conditions.

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An upper bound for the reliability of engineering systems

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Abstract

Since the exact reliability of coherent systems is notoriously difficult to calculate, so bounds can be useful in such situations. In this note, we establish an upper bound for the reliability of coherent systems lifetime by using the system signature. An advantage of the new bound is that it is easy to compute.

Keywords: Coherent system, System signature, Variance.

1 Introduction

It is hard to obtain the exact reliability of coherent systems via straightforward computations in some situations. For instance, it often requires the distribution of component lifetimes as well as the exact structure function of the system. Therefore, there have been efforts made to obtain useful bounds for them. The problem typically consists of computing the possible upper and lower bounds on the reliability, assuming that the lifetime distribution belongs to a common family e.g. IFR, IFRA, NBU. Various results in this area are available in Barlow and Proschan [3]. Since Markov's fundamental inequality, a number of improvements have been obtained under additional assumptions on the underlying distribution function. It states that for a nonnegative random

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variable with mean $\mu = \mathbb{E}(X)$, and survival probability $\overline{F}(t) = P(X > t)$ it holds that $\overline{F}(t) \leq \mu/t$ for $t > \mu$ and $\overline{F}(t) \leq 1$ for $0 < t \leq \mu$, see e.g. Haines and Singpurwalla [1] and Marshall [2]. The aim of this paper is to provide an upper bound based on the following lemma.

Lemma 1.1. *Let X be a random variable with the cumulative distribution function (CDF) $F(x)$. If $\mathbb{E}(X) = 0$ and $\mathbb{E}(X^2) = \sigma^2(X) < \infty$, then*

$$\overline{F}(t) \leq \frac{\sigma^2(X)}{\sigma^2(X) + t^2},\ t > 0.$$

Proof. For all $b > 0$, we have

$$\begin{aligned} P(X > t) &= P(X + b > t + b) \\ &\leq P((X + b)^2 > (t + b)^2) \\ &\leq \frac{E(X + b)^2}{(t + b)^2} \\ &= \frac{\sigma^2(X) + b^2}{(t + b)^2}. \end{aligned}$$

The last inequality is obtained by using the Markov's inequality. The minimum of the right-hand-side of the above relation is attained when $b = \frac{\sigma^2}{t}$, and the proof is then completed. □

Based on the upper bound given in Lemma 1.1, we provide a new upper bound for the reliability function of a coherent system by using the concept of system signature. It depends on the mean and variance of the signature vector as well as the parent distribution function.

2 Main results

Hereafter, we provide an upper bound for the reliability function of the coherent systems. For this purpose, let us consider an increasing nonnegative differentiable function $\psi(x)$ such that $\psi'(x) = \phi(x) \geq 0$. Applying Lemma 1.1, we obtain the following theorem.

Theorem 2.1. *Let X be a lifetime with CDF $F(x)$, and with $\mu_\psi = \mathbb{E}[\psi(X)] < \infty$ and $\psi(t)$ being an increasing function of t. Then,*

$$\overline{F}(t) \leq \begin{cases} \frac{\sigma^2_\psi(X)}{\sigma^2_\psi(X) + (\psi(t) - \mu_\psi)^2}, & \psi(t) > \mu_\psi \\ 1, & \psi(t) \leq \mu_\psi \end{cases} \tag{1}$$

where $\sigma^2_\psi(X) = \sigma^2(\psi(X))$.

Proof. Since $\psi(\cdot)$ is an increasing function, we have

$$\begin{aligned} P(X > t) &= P(\psi(X) > \psi(t)) \\ &= P(\psi(X) - \mu_\psi > \psi(t) - \mu_\psi) \\ &\leq P(Z^2_\psi > (\psi(t) - \mu_\psi)^2) \\ &\leq \frac{\sigma^2_\psi(X)}{\sigma^2_\psi(X) + (\psi(t) - \mu_\psi)^2},\ \psi(t) > \mu_\psi. \end{aligned}$$

The required inequality is obtained by noting that $Z_\psi = \psi(X) - \mu_\psi$ has $\mathbb{E}(Z_\psi) = 0$ and $\mathbb{E}(Z^2_\psi) = \sigma^2_\psi(X)$, and then using Lemma 1.1. Hence, the theorem. □

In particular, if we take $\psi(t) = t$, we simply have $\mu = \mathbb{E}[X]$. Then, we deduce the bound

$$\overline{F}(t) \leq \begin{cases} \frac{\sigma^2(X)}{\sigma^2(X)+(t-\mu)^2}, & t > \mu \\ 1, & t \leq \mu \end{cases} \tag{2}$$

A system is said to be coherent if it does not have any irrelevant components and its structure-function is monotone. A special case of coherent systems is the k-out-of-n system, where the system fails when the k-th component failure occurs. Let T denote the lifetime of a mixed system consisting of n independent and identically distributed (i.i.d.) components with lifetimes $X_1, \ldots, X_n$ having an absolutely continuous CDF F. Denote by $X_{1:n}, \ldots, X_{n:n}$ the ordered lifetimes corresponding to $X_1, \ldots, X_n$ (i.e. the lifetimes of k-out-of-n systems). It is well known that the survival function of a coherent system $\overline{F}_T$ satisfies (see, e.g., Samaniego [4])

$$\overline{F}_T(t) = P(T > t) = \sum_{i=1}^{n} s_i \overline{F}_{i:n}(t), \tag{3}$$

where $\overline{F}_{i:n}(t) = \sum_{j=0}^{i-1} \binom{n}{j} [F(t)]^j [\overline{F}(t)]^{n-j}$ for $i = 1, \ldots, n$, are the survival functions of $X_{i:n}$ ordered component lifetimes. The vector of coefficients $\mathbf{s} = (s_1, \ldots, s_n)$ in (3) is called *system signature* in the literature. Notice that $s_1, \ldots, s_n$ are nonnegative real numbers which do not depend on the common CDF F and such that $\sum_{i=1}^{n} s_i = 1$. One can see that the survival function of a coherent system is a mixture of the survival functions of the i-out-of-n systems with weights s_i. In the following theorem, we have obtained an upper bound for the reliability of coherent systems in terms of the system signature and the parent distribution function of a component lifetime.

Theorem 2.2. *Let T be the lifetime of a coherent system with known signature $\boldsymbol{s}$ consisting of n i.i.d. component lifetimes $X_1, \ldots, X_n$ having common CDF $F(x)$ and PDF $f(x)$. Then,*

$$\overline{F}_T(t) \leq \begin{cases} \frac{(n+1)\left(\sigma^2(\mathbf{s})+\mathbb{E}(\mathbf{s})\right)-(\mathbb{E}(\mathbf{s}))^2}{(n+1)(\sigma^2(\mathbf{s})+\mathbb{E}(\mathbf{s}))-(\mathbb{E}(\mathbf{s}))^2+(n+2)[(n+1)F(t)-\mathbb{E}(\boldsymbol{s})]^2}, & t > F^{-1}\left(\frac{\mathbb{E}(\boldsymbol{s})}{n+1}\right) \\ 1, & t \leq F^{-1}\left(\frac{\mathbb{E}(\boldsymbol{s})}{n+1}\right) \end{cases} \tag{4}$$

where $\mathbb{E}(\boldsymbol{s})$ and $\sigma^2(\mathbf{s})$ denote the mean and variance of the system signature, respectively, and $F^{-1}(u) = \inf\{x; F(x) \geq u\},\ 0 < u < 1$ is the quantile function of cdf $F(x)$.

Proof. By setting $\psi(t) = F(t)$ so that $\phi(t) = f(t)$ and then using Theorem 2.1, we get

$$\overline{F}_T(t) \leq \begin{cases} \frac{\sigma^2(V)}{\sigma^2(V)+(F(t)-\mathbb{E}(V))^2}, & t > F^{-1}\left(\frac{\mathbb{E}(\mathbf{s})}{n+1}\right) \\ 1, & t \leq F^{-1}\left(\frac{\mathbb{E}(\mathbf{s})}{n+1}\right) \end{cases} \tag{5}$$

It is now easy to see that

$$(n+1)^2(n+2)\sigma^2(V) = (n+1)\left(\sigma^2(\mathbf{s}) + \mathbb{E}(\mathbf{s})\right) - (\mathbb{E}(\mathbf{s}))^2,$$

using which the required result follows. □

It can be seen that the upper bound in Theorem 2.2 depends on the mean and variance of the number of components that have failed at the time of system failure and the component lifetime distribution. It is clear that the given bounds are easy to compute for any coherent system. The following example provides an illustration.

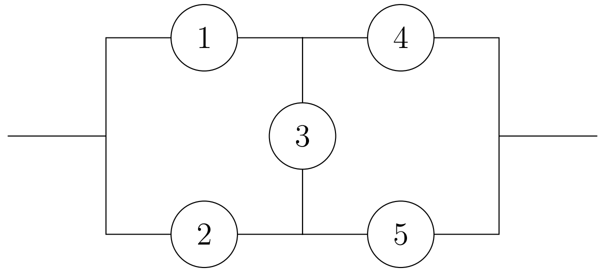

Figure 1: The bridge system with signature $\mathbf{s} = (0, 0.2, 0.6, 0.2, 0)$.

Example 2.3. Let us consider the bridge system shown in Figure 1. It is known that the signature of this system is $\mathbf{s} = (0, 0.2, 0.6, 0.2, 0)$. Let us assume that lifetimes of the component coming from the Weibull model with the CDF

$$F(t) = 1 - e^{-t^{\alpha}}, \quad t > 0, \ \alpha > 0.$$

In this case, we have $F^{-1}(0.5) = \sqrt[\alpha]{0.69314}$, and so the upper bound from Theorem 2.2 is

$$\overline{F}_T(t) \leq \begin{cases} \frac{11.40}{32.40 - 42e^{-t^{\alpha}}}, & t > \sqrt[\alpha]{0.69314} \\ 1, & t \leq \sqrt[\alpha]{0.69314} \end{cases} \tag{6}$$

Moreover, the general upper bound from (2) is

$$\overline{F}_T(t) \leq \begin{cases} \frac{\sigma^2(T)}{\sigma^2(T) + (t - \mathbb{E}(T))^2}, & t > \mathbb{E}(T) \\ 1, & t \leq \mathbb{E}(T) \end{cases} \tag{7}$$

Values of these two bounds, for various α, are plotted in Fig. 2. We observe that for all values of α, the upper bound given in (6) is suitable for the initial values however, when the time increases, the upper bound (7) is better.

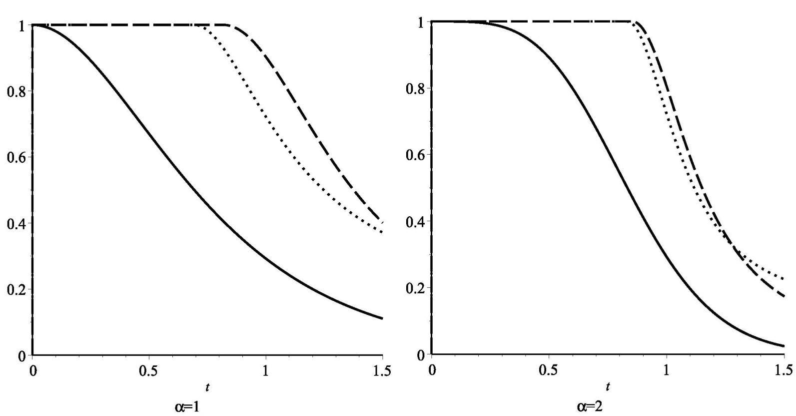

Figure 2: The upper bound given in (7) bound (dashed line) and the new upper bound (dotted line) as well as the exact reliability function of a bridge system (solid line) for the case of Weibull distribution with the shape parameter α.

An immediate consequence of Theorem 2.2 is as follows.

Theorem 2.4. *If $X_{k:n}$ denotes the lifetime of a k-out-of-n system consisting of n I.I.D. component lifetimes $X_1, \ldots, X_n$ having common CDF $F(x)$ and PDF $f(x)$, then we have*

$$\overline{F}_{k:n}(t) \leq \begin{cases} \frac{(n+1)k-k^2}{(n+1)k-k^2-(n+2)[(n+1)F(t)-k]^2}, & t > F^{-1}\left(\frac{k}{n+1}\right) \\ 1, & t \leq F^{-1}\left(\frac{k}{n+1}\right) \end{cases} \tag{8}$$

3 Conclusion

Reliability analysis is a crucial aspect of engineering design, where the ability of a system or component to perform its intended function over time is a critical factor in ensuring safety and efficiency. In this paper, we propose a novel method for obtaining upper bounds for the reliability of a random lifetime, which offers significant advantages over existing approaches. Our method produces algebraic inequalities that are more practical and easier to estimate from data than those based on upper tail moments. The given bound provides a powerful tool for assessing the reliability of complex systems with diverse components and failure modes.

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Quantile based dynamic cumulative extropy

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Abstract

In this paper, we propose the quantile based dynamic cumulative extropy in residual lifetime. Various characterizations are obtained based on the lifetime distributions and quantile-based reliability measures function. We introduce a new stochastic order based on the quantiles that is built on this measure. Also, some properties of the quantile based dynamic cumulative extropy is studied.

Keywords: Characterization, Quantile Function, Residual Extropy, Stochastic Orders, Survival Extropy.

1 Introduction

The extropy measure studied here to give a quantified measure of uncertainty involved in a random variable has become a widely used measure. Various areas such as actuarial sciences, survival analysis and reliability analysis have been profiting this measure through many applications. Extropy of a non-negative random variable X which is absolutely continuous and has the probability density function (pdf) $f_X(x)$ is defined by [6] as a complement dual of Shannon entropy. This is a measure of uncertainty related to the outside ([1]) and is defined as

$$J(X) = -\frac{1}{2}\int_0^\infty f_X^2(x)dx = -\frac{1}{2}E\big(f_X(X)\big), \tag{1}$$

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where E denotes the expected value operator.
Since we can not use $J(X)$ in the context of a random variable which has already survived for a period of time, [12] suggested the measure of residual extropy, which is defined as

$$J(X;t) = -\frac{1}{2}\int_t^{\infty} \frac{f_X^2(x)}{\bar{F}_X^2(t)}dx, \tag{2}$$

in which, $f_X(x)$ and $\bar{F}_X(x)$ respectively are pdf and survival functions of the random variable X. Many interesting properties of extropy have been found by scholars. For instance, [11], [3], [14], [16] studies and etc.
In this direction, [2] and [16] explored an expression of the cumulative extropy as

$$J_s(X) = -\frac{1}{2}\int_0^{\infty} \bar{F}_X^2(x)dx. \tag{3}$$

A probability distribution function can be specified in two ways; as a distribution function or as a quantile function. The quantile function of random variable X would be defined as

$$Q_X(p) = F_X^{-1}(p) = \inf\{x : F_X(x) \geq p\},\ 0 \leq p \leq 1. \tag{4}$$

Equivalently, $F_X(Q_X(p)) = p$, that implies $f_X(Q_X(p))q_X(p) = 1$, where $f_X(Q_X(p))$ and $q_X(p) = \frac{d}{dp}Q_X(p)$ respectively denote the density quantile function and quantile density function (qdf) ([10]).
The hazard rate $h_X(x)$ or its equivalent the hazard quantile function (hqf) are the primary concepts to present the physical properties of failure patterns (see [7]) defined as

$$H_X(p) = h_X(Q_X(p)) = \frac{f_X(Q_X(p))}{\bar{F}_X(Q_X(p))} = \frac{1}{(1-p)q_X(p)}. \tag{5}$$

The hqf can be interpreted as the explanation of the conditional probability of a case of failure within the next small time interval if we have the survival until $100(1-u)\%$ point of distribution. We also can determine the quantile function by the hqf as

$$Q_X(p) = \int_0^p \frac{du}{(1-u)H_X(u)}, \quad p \in (0,1). \tag{6}$$

Corresponding to [7] we would have mean residual quantile function (mrqf) as

$$M_X(p) = \frac{1}{1-p}\int_p^1 \big(Q_X(u) - Q_X(p)\big)du \tag{7}$$

$$= \frac{1}{1-p}\int_p^1 \frac{du}{H_X(u)}, \quad \forall p \in (0,1). \tag{8}$$

Similar to [5] the mrqf also uniquely determines the quantile function by

$$Q_X(p) = \int_0^p \frac{M_X(u)}{1-u}du - M_X(p) - M_X(0), \quad p \in (0,1). \tag{9}$$

Equivalently

$$q_X(p) = \frac{M_X(p)}{(1-p)} - M_X'(p), \tag{10}$$

where $M_X'(p) = \frac{d}{dp}M_X(p)$.

The rest of this article is structured as follows. Section 2, we provide some properties and characterizations of the quantile-based dynamic cumulative extropy in residual lifetime (QDCEXR). Section 3, includes discussion and our conclusions.

2 Main results

The length of study time has been considered as a main variable of interest in many fields of statistical studies such as reliability, survival analysis, economic, business etc. For example see the recently study by [9]. The information measures in such cases are time-dependent, thus they are dynamic.

The concept of extropy in statistical applications is usually applied to score the forecasting distributions based on the total log scoring method. For example, under the total log scoring rule, the negative sum of the entropy and extropy equals the expected score of a forecasting distribution [2, 4]. Inspired by the dynamic cumulative entropy and considering a parallel (series) system with independent and identical components with lifetimes $X_1, ..., X_n$ with common cdf has been studied by various researchers. For example we can refer to [2, 8, 16].

In this section, we propose the dynamic cumulative extropy residual lifetime.

The dynamic cumulative version of $J(X)$ for residual lifetime t is given by

$$J_s(X,t) = -\frac{1}{2}\int_t^{\infty} \frac{\bar{F}^2(x)}{\bar{F}^2(t)}dx, \quad x \geq 0. \tag{11}$$

Then the quantile-based dynamic cumulative residual extropy becomes

$$J_s(X,p) = -\frac{1}{2}\int_p^{1} \frac{(1-u)^2}{(1-p)^2} q_X(u)du, \quad 0 < p < 1. \tag{12}$$

Differentiating both sides of (12) with respect to p, we get

$$q_X(p) = -\frac{4J_s(X,p)}{1-p} + 2J_s^{'}(X,p), \tag{13}$$

where $J_s^{'}(X,p) = \frac{d}{dp}J_s(X,p)$. Then $J_s(X,p)$ uniquely determines the distribution function.

In term of hqf $J_s(X,p)$ reduces to

$$J_s(X,p) = -\frac{1}{2}\int_p^{1} \frac{(1-u)}{(1-p)^2}\frac{1}{H_X(u)}du, \tag{14}$$

Also, $J_s(X,p)$ can be expressed in term of the mrqf

$$J_s(X,p) = -\frac{1}{2}M_X(p) + \frac{1}{2(1-p)^2}\int_p^{1}(1-u)M_X(u)du, \tag{15}$$

Theorem 2.1. *The following relations are established*

$$a) \quad J_s(X,p) = \frac{-k}{2(1-p)^2}\bar{B}(p,\gamma+1,3-(A+\gamma)),$$

$$b) \quad J_s(X,p) = \frac{-k}{2(1-p)^2}\bar{\Gamma}(p,\gamma+1,3-A).$$

if and only if

$$a) \quad q_X(u) = ku^{\gamma}(1-u)^{-(A+\gamma)}, \quad k>0,$$

$$b) \quad q_X(u) = k(1-u)^{-A}\big(-\ln(1-u)\big)^{-\gamma}. \tag{16}$$

where A and γ are real constants.

Proof. a) We have

$$J_s(X,p) = \frac{-k}{2(1-p)^2}\int_p^1 u^\gamma (1-u)^{2-(A+\gamma)}du = \frac{-k}{2(1-p)^2}\bar{B}\big(p, \gamma+1, 3-(A+\gamma)\big).$$

b) We have

$$\begin{aligned} J_s(X,p) &= \frac{-k}{2(1-p)^2}\int_{-\ln(1-p)}^{\infty} e^{-(3-A)z} z^\gamma dz \\ &= \frac{-k}{2(1-p)^2}\bar{\Gamma}\big(-\ln(1-p), \gamma+1, 3-A\big). \end{aligned}$$

□

Figure 1 gives the graphs of $J_s(X,p)$ for $k=2$, $A=0,1$ and $\gamma=-2,-1,0,1,(1.5,2)$ respectively. Note that $J_s(X,p)$ is non-decreasing for $\gamma<0$, constant for $\gamma=0$, non-increasing for $\gamma>0$. Similarly, Figure 2 shows the graphs of $J_s(X,p)$ for $k=2$, $A=0,1$ and $\gamma=-2,-1,0,1,2$. It is

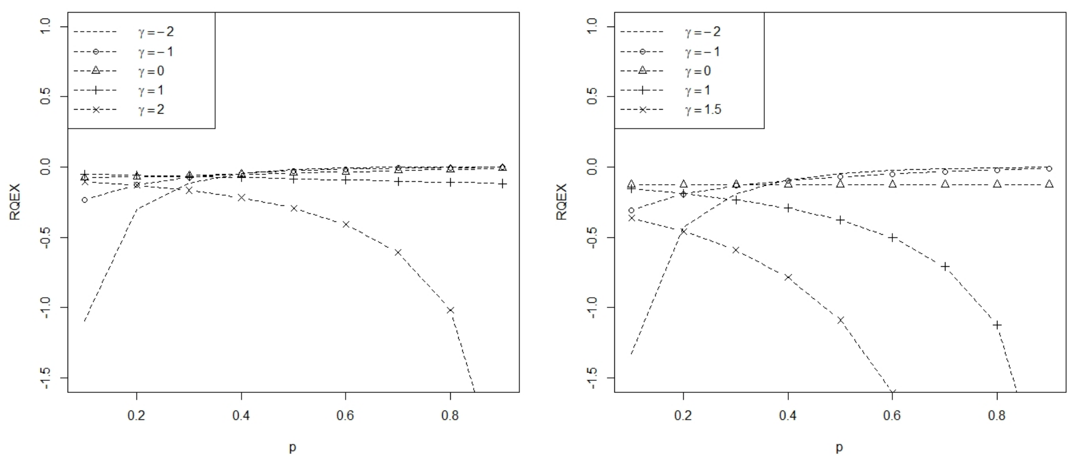

Figure 1: Graphs of $J_s(X,p)$ for $A=0$, $\gamma=-2,-1,0,1,2$ (Left panel), and $A=1$, $\gamma=-2,-1,0,1,1.5$ (Right panel), in Theorem 2.1 (a).

shown that $J_s(X,p)$ is non-decreasing , constant for $\gamma=0$ and non-increasing for $\gamma<0$, $\gamma=0$ and $\gamma>0$ respectively.

Remark 2.2. There are some important distribution functions within the family of distributions such as exponential ($\gamma=0, A=\frac{a}{a+1}+1$), Power($\gamma=\frac{1}{\beta}-1, A=0$), Govindarajulu ($\beta-1, A=-1$). Similarly (16) contains several well known distributions which include Weibull distribution ($A=1, \gamma=\frac{1}{a}-1$), Pareto ($A>1, \gamma=0$) and rescaled Beta ($A<1, \gamma=0$).

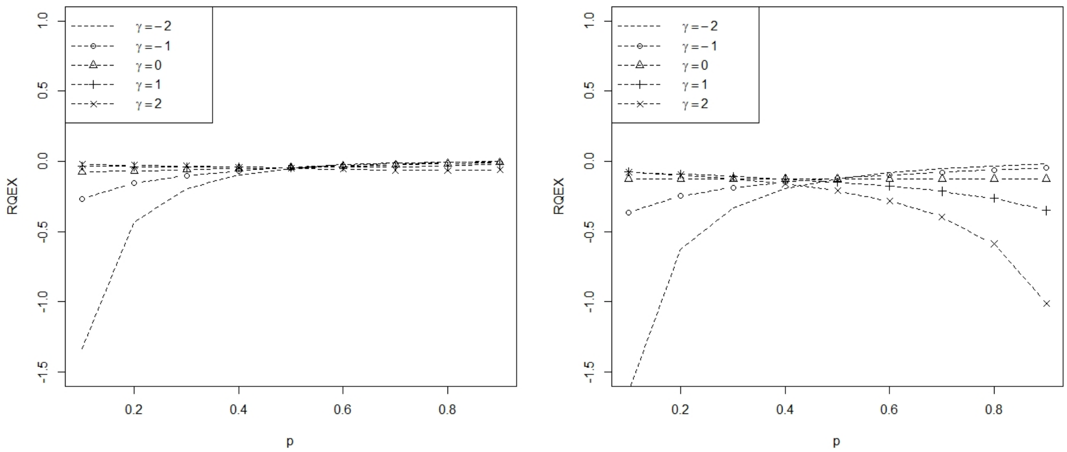

Figure 2: Graphs of $J_s(X,p)$ for $A=0$ (Left panel), and $A=1$ (Right panel) and $\gamma = -2,-1,0,1,2$ in Theorem 2.1 (b).

Table 1 represents the quantile function and cumulative residual extropy for some distributions.

Table 1. Quantile function and cumulative residual extropy for some distributions

Distribution	$Q(p)$	$J_s(X,p)$
Exponential	$-\frac{\log(1-p)}{\lambda}$	$-\frac{1}{4\lambda}$
Pareto II	$\gamma\big((1-p)^{\frac{-1}{c}}-1\big)$	$-\frac{\gamma}{2(2c-1)}(1-p)^{-\frac{1}{c}}$
Rescaled Beta	$R\big(1-(1-p)^{\frac{1}{c}}\big)$	$\frac{-R}{2(1+2c)}(1-p)^{\frac{1}{c}}$
Generalized Pareto	$\frac{b}{a}[(1-p)^{\frac{-a}{a+1}}-1]$	$-\frac{b}{2(a+2)}(1-p)^{-\frac{a}{a+1}}$
Power	$\gamma p^{\frac{1}{\beta}}$	$\frac{-\gamma}{2(1-p)^2}\left[\frac{2\beta^2}{(1+\beta)(2\beta+1)}-p^{\frac{1}{\beta}}\left(1+\frac{p^2}{2\beta+1}-\frac{2p}{1+\beta}\right)\right]$
Uniform	$a+(b-a)p$	$-\frac{(b-a)(1-p)}{6}$
Davis	$\frac{cp^{\lambda_1}}{(1-p)^{\lambda_2}}$	$\frac{-c}{2(1-p)^2}\left[\lambda_1\bar{B}(p,\lambda_1,3-\lambda_2)+\lambda_2\bar{B}(p,1+\lambda_1,2-\lambda_2)\right]$
Skew Lambda	$\gamma p^{\lambda}-(1-p)^{\lambda}$	$\frac{-\lambda}{2(1-p)^2}\left[\gamma\bar{B}(p,\lambda,3)+\bar{B}(p,1,2+\lambda)\right]$
Weibull	$(\frac{-\ln(1-p)}{\lambda})^c$	$\frac{-2c}{\lambda(1-p)^2}\bar{\Gamma}(p,1+c,2)$
Govindarajulu	$\gamma\big((\beta+1)p^{\beta}-\beta p^{\beta+1}\big)$	$-\frac{\gamma\beta(\beta+1)}{2(1-p)^2}\left[\bar{B}(p,\beta,3)-\bar{B}(p,\beta+1,3)\right]$

Identifying different classes of probability models in terms of monotonous behaviors of the uncertainty measures would by usually useful. Hence, we define the following non-parametric classes based on $J_s(X,p)$.

Definition 2.3. X is said to have increasing(decreasing) quantile dynamic cumulative extropy in residual lifetime which are abbreviated as IQDCEXR (DQDCEXR) if $J_s(X,p)$ is increasing (decreasing) in p.

Now, we can find from the relation (13) that if X is IQDCEXR (DQDCEXR), then $J_s(X,p) \geq (\leq) \frac{-q_X(p)(1-p)}{4}$. It follows that if X is IQDCEXR (DQDCEXR), then $J_s(X,p) \geq (\leq) -\frac{1}{4H_X(p)}$. Note that for the exponential distribution, $J_s(X,p) = -\frac{1}{4H_X(p)}$. Thus exponential distribution is the boundary of IQDCEXR and DQDCEXR classes.

Theorem 2.4. *Let X be IQDCEXR (DQDCEXR), then $J_s(X,p) \geq (\leq) -\frac{1}{4H_X(p)}$, which provide upper (lower) bounds for $J_s(X,p)$ with respect to IQDCEXR (DQDCEXR).*

Proof. Differentiating $J_s(X,p)$ in (14) with respect to p we can obtain

$$J_s^{'}(X,p) = \frac{1}{2}\frac{1}{(1-p)H_X(p)} + \frac{2J_s(X,p)}{1-p}.$$

Since X is IQDCEXR (DQDCEXR) we would have $J_s(X,p) \geq (\leq) -\frac{1}{4H_X(p)}$. □

Theorem 2.5. *$J_s(X,p) = c$ is a constant if and only if X be a random variable with exponential distribution.*

Proof. Assume that $J_s(X,p) = c$. Using (13) we obtain $Q_X(p) = 4c\ln(1-p) = -\frac{\ln(1-p)}{\lambda}$, where $\lambda = \frac{-1}{4c}$.
On the opposite side, suppose that X follows exponential distribution with quantile function $Q_X(p) = -\frac{\ln(1-p)}{\lambda}$. Now from (12) we have $J_s(X,p) = \frac{-1}{4\lambda} = c$ is a constant. □

Theorem 2.6. *If X is continuous random variable with $F_X(x)$ and $f_X(x)$ as the cdf and pdf respectively and with IQDCEXR property, also $\phi(.)$ is a non-negative increasing and convex function, then $\phi(X)$ is IQDCEXR.*

Proof. Let $Y = \phi(X)$ with cdf and pdf $G_Y(y)$ and $g_Y(y)$ respectively. Then

$$g_Y(y) = \frac{f_X(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))} = \frac{1}{\phi'(Q_X(u))q_X(u)}$$

.
Furthermore,

$$\begin{aligned} J_s(Y,p) &= -\frac{1}{2}\int_p^1 \frac{(1-u)^2}{(1-p)^2} q_Y(u)du \\ &= -\frac{1}{2}\int_p^1 \frac{(1-u)^2}{(1-p)^2}\big(\phi^{'}(Q_X(u)) \pm 1\big) q_X(u)du \\ &= J_s(X,p) + \frac{1}{2}\int_p^1 \frac{(1-u)^2}{(1-p)^2}\big(1-\phi^{'}(Q_X(u))\big) q_X(u)du. \end{aligned}$$

Since ϕ is a convex, $\phi^{'}(Q_X(p)) < \phi^{'}(Q_X(u)), \quad 0 < p < u < 1$, that yields

$$\begin{aligned} J_s(Y,p) &< J_s(X,p) + \frac{1}{2}\frac{\big(1-\phi^{'}(Q_X(p))\big)}{(1-p)^2}\int_p^1 (1-u)^2 q_X(u)du \\ &= J_s(X,p) - J_s(X,p)\big(1-\phi^{'}(Q_X(p))\big) \\ &= J_s(X,p)\phi^{'}(Q_X(p)), \end{aligned}$$

which complete the proof. □

Example 2.7. Let X has rescaled beta distribution with $Q_X(u) = R\big(1-(1-u)^{\frac{1}{c}}\big)$ when $Y = X^{\beta}, \quad \beta > 0$, with quantile function $Q_Y(u) = R^{\beta}\big(1-(1-u)^{\frac{1}{c}}\big)^{\beta}$. therefore, using the above if X is IQDCEXR then Y is also IQDCEXR.

Theorem 2.8. *Let X be a non-negative absolutely continuous random variable with $J_s(X,p) = c.M_X(p)$. Then*

$$Q_X(p) = k\frac{c}{2c+0.5}\left(1-(1-p)^{\frac{-c}{2c+0.5}}\right).$$

Proof. From (15) we would have

$$M_X'(p) = M_X(p)\frac{2c+0.5}{(1-p)(c+.5)},$$

which on simplification resolve to $M_X(p) = k(1-p)^{-\frac{2c+0.5}{c+.5}}$ and substituting it in (9), we obtain the result. □

Based on Theorem 2.8, we show that the power function distribution can be characterized using $J_s(X,p)$.

Remark 2.9. Let X be a non-negative continuous random variable with the quantile function $Q(.)$ and mean residual quantile function $M(.)$. Then

$$J_s(X,p) = -\frac{1}{2}\frac{\beta+1}{2\beta+1}M_X(p),$$

if and only if X has power function distribution.

Theorem 2.10. *Let X be a non-negative continuous random variable such that $J_s(X,p) = -\frac{c(p)}{2}M_X(p)$ then*

$$M_X(p) = \frac{e^{\int_0^p \frac{du}{(1-u)\big(c(u)-1\big)}}}{(1-p)^2\big(c(p)-1\big)}. \tag{17}$$

Proof. Assuming the condition $J_s(X,p) = -\frac{c(p)}{2}M_X(p)$ and taking the definition (12) yields

$$\int_p^1 \frac{(1-u)^2}{(1-p)^2}q(u)du = c(p)M_X(p). \tag{18}$$

Differentiating with respect to p we obtain

$$(1-p)^2 q(p) = 2(1-p)c(p)M_X(p) - c^{'}(p)M_X(p)(1-p)^2 - (1-p)^2 c(p)M_X^{'}(p). \tag{19}$$

So from (6) we can write

$$M_X(p)\big(2c(p) - 1 - (1-p)c^{'}(p)\big) = M_X^{'}(p)(1-p)\big(c(p)-1\big). \tag{20}$$

On solving the first order differential equation in $M_X(p)$ we get (17). $\square$

Generally, various stochastic orders could be implemented for comparison of two random variables. We now consider the hazard quantile function order to compare two random variables based on $J_s(X,p)$.

Definition 2.11. X is said to be smaller than Y in QDCEXR ($X \overset{QDCEXR}{\leq} Y$) if $J_s(X,p) \leq J_s(Y,p)$, $\forall p \in (0,1)$.

Theorem 2.12. *Let $X \overset{HQ}{\leq} Y$, then $X \overset{QDCEXR}{\geq} Y$.*

Proof. Assuming $X \overset{HQ}{\leq} Y$ implies $(1-p)q_X(p) \leq (1-p)q_Y(p)$. So, in glimpse, from (12) it yields $J_s(X,p) \geq J_s(Y,p)$. $\square$

Theorem 2.13. *Let X and Y be two random variables such that $X \overset{QDCEXR}{\leq} Y$. Then for a non-negative convex function $\phi(.)$, $\phi(X) \overset{QDCEXR}{\geq} \phi(Y)$.*

Proof. It is enough to show that

$$\int_p^1 \frac{(1-u)^2\phi^{'}\big(Q_X(u)\big)q_X(u)}{(1-p)^2}du \leq \int_p^1 \frac{(1-u)^2\phi^{'}\big(Q_Y(u)\big)q_Y(u)}{(1-p)^2}du.$$

Since $X \overset{QDCEXR}{\leq} Y$ we have $J_s(X,p) \leq J_s(Y,p)$, which is equivalent to

$$\int_p^1 (1-u)^2 q_X(u)du \geq \int_p^1 (1-u)^2 q_Y(u)du.$$

Thus the function is increasing in p, $q_X(p) \leq q_Y(p)$. Since ϕ is convex function: $\phi^{'}\big(Q_X(u)\big) \leq \phi^{'}\big(Q_Y(u)\big)$. Hence we would have $\phi(X) \overset{QDCEXR}{\geq} \phi(Y)$. $\square$

We express some of the relationships in the following theorems. Note that $\overset{Disp}{\leq}$, $\overset{HR}{\leq}$ and $\overset{St}{\leq}$ denote the dispersive, the hazard rate and the usual stochastic orders respectively. For compare comprehensive discussion on various consents of stochastic ordering based on reliability measures, see [13].

Theorem 2.14. *If $X \overset{MQ}{\leq} Y$ then $X \overset{QDCEXR}{\geq} Y$.*

Proof. It is obvious that, $X \overset{MQ}{\leq} Y$, thus $X \overset{HQ}{\leq} Y$, So Theorem 2.12 completes the proof. $\square$

Lemma 2.15. *For two continuous non-negative random variable X and Y,*

- $X \overset{Disp}{\leq} Y \Leftrightarrow X \overset{HQ}{\leq} Y$
- *If X or Y is decreasing (increasing) hazard rate, then $X \overset{HR}{\leq} Y \Rightarrow (\Leftarrow) X \overset{HQ}{\leq} Y$*

- *If X and Y have the same lower end of support and if $\frac{Q_Y(p)}{Q_X(p)}$ is increasing in $p \in (0,1)$ then* $X \overset{St}{\leq} Y \Rightarrow X \overset{HQ}{\leq} Y$

Theorem 2.16. *Let X and Y be two non-negative random variables having continuous quantile density function $q_X(p)$ and $q_Y(p)$ and quantile functions $Q_X(p)$ and $Q_Y(p)$ respectively. then*

- *If $X \overset{Disp}{\leq} Y$, then $X \overset{QDCEXR}{\geq} Y$.*
- *If X or Y is DHR, then if $X \overset{HR}{\leq} Y$, then $X \overset{QDCEXR}{\geq} Y$.*
- *If X and Y have the same lower end of the support and if $\frac{Q_Y(p)}{Q_X(p)}$ is increasing in $p \in (0,1)$, then if $X \overset{St}{\leq} Y$, yields $X \overset{QDCEXR}{\geq} Y$.*

Proof. Using Lemma 2.15 and Theorem 2.12 the proof is complete. □

3 Conclusion

This study has introduced quantile-based in cumulative residual extropy. It is shown that the quantile-based cumulative extropy in residual lifetime determine the distribution through an explicit expression in a unique way. We also investigated some characteristics and properties of these quantile-based measures in the context of the important the quantile-based survival functions such as hazard and mean residual functions. Also some characterizes on stochastic orders included the quantile form of usual, hazard rate and dispersion rate orders, besides, proposed some quantile-based orders on residual quantile cumulative extropy have studied.

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An optimal selection problem in a k-out-of-n system with dependent components

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Abstract

One of the methods to improve the reliability of a system is redundancy allocation. In this paper, we aim to examine the redundancy allocation in a k-out-of-n system with dependent and heterogeneous components. Some examples are provided to illustrate the optimal allocation based on criteria of the usual stochastic order and the mean residual lifetime of the system.

Keywords: Reliability, Active redundancy, Copula function.

1 Introduction

The redundancy problem is one of the most important issues in reliability engineering to increase the reliability of systems. There are two types of redundancy: active, and cold-standby redundancy. In the active mode, the redundant component works with other components of the system while in the cold standby, the spare starts working as soon as the corresponding component fails. It is important to indicate the spares are allocated to which components of the system such that system

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reliability is maximized. Many researchers have studied the redundancy allocation problem (RAP) for systems with independent components; see e.g., [3, 10, 11, 8]. Boland et al. [3] showed that based on stochastic ordering for a k-out-of-n system having independent components, a redundant component should be allocated to the weakest component. Ding and Li [5] explored the allocation problem of active redundancies to a k-out-of-n system with i.i.d. components based on the hazard rate order. In [1], the RAP in a k-out-of-n system was considered under the assumption that both types of active and standby redundant components are used in the system. Zhang [13] explored, by using stochastic orders, the optimal allocation of active redundancies to a weighted k-out-of-n system with independent components.

In the aforementioned works, the lifetimes of system components are assumed to be independent. There are a few studies on the RAP of systems with dependent components. In [6], the redundancy allocation was studied for the series systems with dependent component lifetimes and one active (or, cold standby) redundant component. This work was extended to the case where two spares are applied; see, [7]. For more studies, one can refer to [12, 4, 2]. In this paper, we investigate the problem of allocation of one active redundant component for a k-out-of-n system having dependent components. The problem is also examined for two redundant components. The proofs of theorems are omitted because of restrictions in page numbers.

2 Main results

Consider a k-out-of-n system with dependent components. It is assumed that there is one spare component for the system. The lifetimes of components and spare can be dependent and follow arbitrary distribution functions (DFs). Let $X_1, \ldots, X_n$ denote the lifetimes of components with DFs $F_1, \ldots, F_n$. Then, the total of active components at time t is $\mathcal{B}(t) = \sum_{i=1}^{n} I(X_i > t)$, and the lifetime of the system is defined as

$$T := \inf\{t : \mathcal{B}(t) < k\}.$$

Then, the reliability function of the system is obtained by

$$P(T > t) = P(\mathcal{B}(t) \geq k).$$

Under these conditions, we want to explore that the active spare component must be allocated to what component to optimize reliability. Before that, we present the definition of usual stochastic order [9].

Definition 2.1. For two non-negative random variables (r.v.s) X and Y with survival functions $\bar{F}_X$ and $\bar{F}_Y$, X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{st} Y$, if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all $t \geq 0$.

Let Y denote the lifetime of active redundant component with DF G. Suppose that T_i, and T_j denote the system lifetimes if the redundant component is allocated to the component i and j, respectively. The considered criterion is that if $T_i \geq_{st} T_j$, then the component i is preferred to the component j for the redundant component allocation.

Theorem 2.2. *Suppose that $Y, X_1, \ldots, X_n$ are dependent r.v.s. Then, $T_i \geq_{st} T_j$ if and only if for all $t \geq 0$*

$$\alpha_i(t) \leq \alpha_j(t), \tag{1}$$

where

$$\alpha_i(t) = P(\mathcal{B}^{-(i,j)}(t) \geq k-2, X_i > t, Y > t) - P(\mathcal{B}^{-(i,j)}(t) \geq k-1, X_i > t, Y > t),$$
$$\alpha_j(t) = P(\mathcal{B}^{-(i,j)}(t) \geq k-2, X_j > t, Y > t) - P(\mathcal{B}^{-(i,j)}(t) \geq k-1, X_j > t, Y > t),$$

in which $\mathcal{B}^{-(i,j)}(t) = \sum_{\substack{\ell=1 \\ \ell \notin \{i,j\}}}^{n} I(X_\ell > t)$.

In the following, we present an example for applying Theorem 2.2. First, it is reminded that $\tilde{H} : [0,1]^d \to [0,1]$ is a d-dimensional copula if it is a joint DF of a d-dimensional random vector with uniform marginal distributions. If H denotes the joint DF of r.v.s $X_1, \ldots, X_d$ with marginal DFs $F_1, \ldots, F_d$, respectively, based on the copula function $\tilde{H}$, we can write

$$H(x_1, \ldots, x_d) = \tilde{H}(F_1(x_1), \ldots, F_d(x_d)).$$

Example 2.3. Consider a 2-out-of-3 system. Assume that the joint DF of $(X_1,\ X_2,\ X_3,\ Y)$ is given, based on the Clayton copula function, as follows

$$H(x_1, x_2, x_3, y) = (\sum_{i=1}^{3} F_i^{-\theta}(x_i) + G^{-\theta}(y) - 3)^{-1/\theta},$$

where $y \geq 0$, $x_i \geq 0$, $i = 1, \ldots, 3$ and $\theta = 0.5$. Suppose that

$$F_1(t) = 1 - \exp(-t), \quad F_2(t) = 1 - (1+t)^{-1/3}, \quad F_3(t) = 1 - (1+t)^{-1},$$

$$G(t) = 1 - \exp(-t^{0.5}).$$

The marginal DFs of X_1, X_2, X_3 and Y are plotted in Figure 1. It shows that $X_1 \leq_{st} X_3 \leq_{st} X_2$ and Figure 2 shows that when the spare is allocated to the weakest component, the system reliability is in the optimal situation.

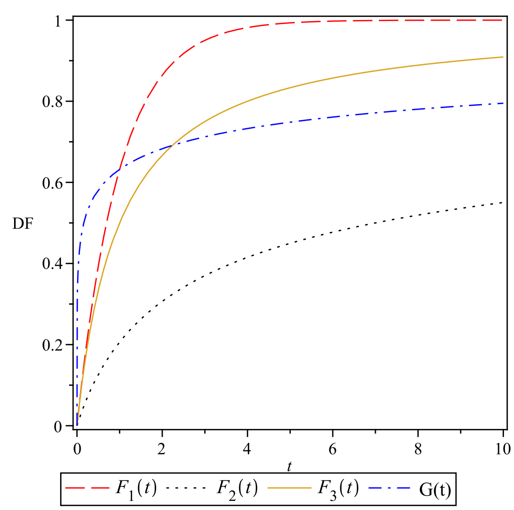

Figure 1: The plot of marginal DFs of main and redundant components.

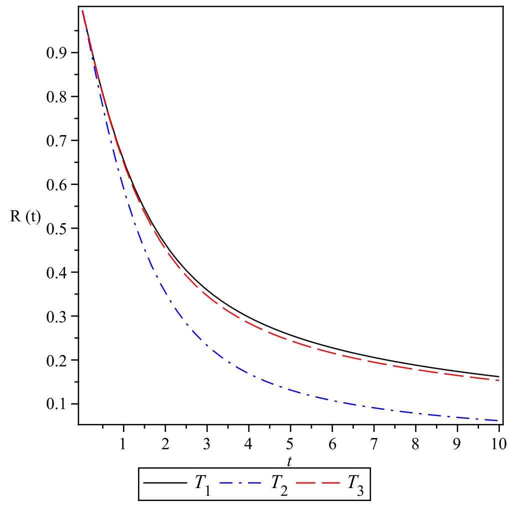

Figure 2: The plot of reliability functions of T_i, $i = 1, 2, 3$.

It is remarkable that in relation (1) the dependency between X_i and X_j has no role. The following proposition achieves a simplification of the condition in Theorem 2.2 for the case when the spare lifetime is independent of components lifetime.

Proposition 2.4. *Suppose that the lifetime of spare Y is independent of the lifetimes of main components of system. Then $T_i \geq_{st} T_j$ if and only if for all $t \geq 0$,*

$$\alpha_i^*(t) \leq \alpha_j^*(t), \tag{2}$$

where

$$\alpha_i^*(t) = P(\mathcal{B}^{-(i,j)}(t) \geq k-2, X_i > t) - P(\mathcal{B}^{-(i,j)}(t) \geq k-1, X_i > t),$$
$$\alpha_j^*(t) = P(\mathcal{B}^{-(i,j)}(t) \geq k-2, X_j > t) - P(\mathcal{B}^{-(i,j)}(t) \geq k-1, X_j > t).$$

The mean residual lifetime (MRL) is a well-known function in the reliability context. The MRL function quantifies the expected remaining life of system when we know it has survived to a certain time point x. Then, for a lifetime variable X with finite expectation, the MRL at time x is defined as $m_X(x) := E(X - x|X > x)$. It is obvious that $m_X(0)$ is equal to the mean of X. For two non-negative r.v.s X, and Y, X is said to be less than Y in MRL concept (denoted by $X \leq_{mrl} \mathrm{Y}$) if and only if $m_X(x) \leq m_Y(x)$ for all $x \geq 0$. It should be noted neither of the orders $\leq_{st}$ and $\leq_{mrl}$ implies the other; counterexamples can be found in the literature [9].

Based on the concept of MRL, it seems that the MRL can be a suitable concept to indicate the optimal selection in RAP. Thus, we derived an algorithm for the problem gotten the amounts of MRL of a k-out-of-n system under the described conditions of this paper. This algorithm is not given in the paper because of restrictions in the page number. In the following, we explore this problem as an example.

Example 2.5. Consider a 3-out-of-4 system. Let the marginal DFs of $X_1, \ldots, X_4$ and Y be $F_1(t) = 1 - \exp(-t^{2/3})$, $F_2(t) = 1 - \exp(-t^{1/2})$, $F_3(t) = 1 - (1+t)^{-1}$, $F_4(t) = 1 - \exp(-t^3)$ and $G(t) = 1 - \exp(-t^{1/5})$, respectively. In Table 1, using simulation, we compute the MRL of T_i, $i = 1, \ldots, 4$ with age $x = 0.8487657$ when Gumbel and Frank copula functions are considered for the joint DF of $(X_1, \ldots, X_4, Y)$. Table 1 shows that if the joint DF of $(X_1, \ldots, X_4, Y)$ is represented by Gumbel copula function with parameter $\alpha = 2.5$, as

$$H(x_1, \ldots, x_4, y) = \exp\{-[\sum_{i=1}^{4}(-\ln F_i(x_i))^\alpha + (-\ln G(y))^\alpha]\}^{1/\alpha},$$

then the MRL of T_4 is equal to 3.41 and is maximal.

Table 1. The maximum value of MRLs of T_i, $i = 1, \ldots, 4$ in a 3-out-of-4 system at time $x = 0.8487657$.

Gumbel copula			Frank copula		
α	Optimal lifetime	MRL	β	Optimal lifetime	MRL
1.5	T_4	2.80	0.5	T_4	0.97
2.0	T_4	3.24	1.0	T_4	1.14
2.5	T_4	3.41	1.5	T_4	1.31

In what follows, we investigate the problem in the situation where two active redundant components with lifetimes Y_1 and Y_2 (with DFs G_1 and G_2, respectively) are allocated to a k-out-of-n system with dependent components. The decision is based on the case where the reliability function is more in the sense of stochastic order [9].

Theorem 2.6. *Suppose that $Y_1, Y_2, X_1, \ldots, X_n$ are dependent r.v.s. Further, let $T_{i,j}$ ($T_{j,i}$) be the lifetime of system when the redundant component with lifetime Y_1 is allocated to the component i (j) and the other redundant component is allocated to the component j (i). Then $T_{i,j} \geq_{st} T_{j,i}$ if and only if for all $t \geq 0$*

$$\beta_i(t) \geq \beta_j(t), \tag{3}$$

where

$$\begin{aligned}\beta_i(t) &= P(\mathcal{B}^{-(i,j)}(t) \geq k-2, X_i > t, Y_2 > t) + P(\mathcal{B}^{-(i,j)}(t) \geq k-1, X_i > t, Y_1 > t)\\ &\quad - P(\mathcal{B}^{-(i,j)}(t) \geq k-1, X_i > t, Y_2 > t) - P(\mathcal{B}^{-(i,j)}(t) \geq k-2, X_i > t, Y_1 > t),\\ \beta_j(t) &= P(\mathcal{B}^{-(i,j)}(t) \geq k-2, X_j > t, Y_2 > t) + P(\mathcal{B}^{-(i,j)}(t) \geq k-1, X_j > t, Y_1 > t)\\ &\quad - P(\mathcal{B}^{-(i,j)}(t) \geq k-1, X_j > t, Y_2 > t) - P(\mathcal{B}^{-(i,j)}(t) \geq k-2, X_j > t, Y_1 > t).\end{aligned}$$

Example 2.7. Consider a 2-out-of-3 system. Suppose that the joint DF of (X_1, X_2, X_3, Y_1, Y_2) is given, based on the Gumbel copula function with parameter $\alpha = 1.2$, as

$$H(x_1, x_2, x_3, y_1, y_2) = \exp\{-[\sum_{i=1}^{3}(-\ln F_i(x_i))^\alpha + \sum_{j=1}^{2}(-\ln G_j(y_j))^\alpha]\}^{1/\alpha}.$$

Further assume that $F_1(t) = \Phi((\ln(t))/2)$, where Φ is the DF of standard normal distribution, $F_2(t) = 1 - \exp(-3t)$, $F_3(t) = 1 - (1+t)^{-1/2}$, $G_1(t) = 1 - (1+t)^{-1/3}$ and $G_2(t) = \Phi((\ln(t))/\sqrt{5})$. Figure 3 represents the plots of marginal DFs of the X_1, X_2, X_3, Y_1 and Y_2. In Figure 3 it is clear that $X_2 \leq_{st} X_1$, $X_2 \leq_{st} X_3$ and $Y_2 \leq_{st} Y_1$. The reliability functions of $T_{i,j}$, $i, j = 1, \ldots, 3$, $i \neq j$ are plotted in Figure 4. It is concluded that the system reliability is in the optimal state if the stronger spare is allocated to the weakest component and weaker spare is allocated to the component with lifetime X_1.

From Theorem 2.6, the following proposition is concluded.

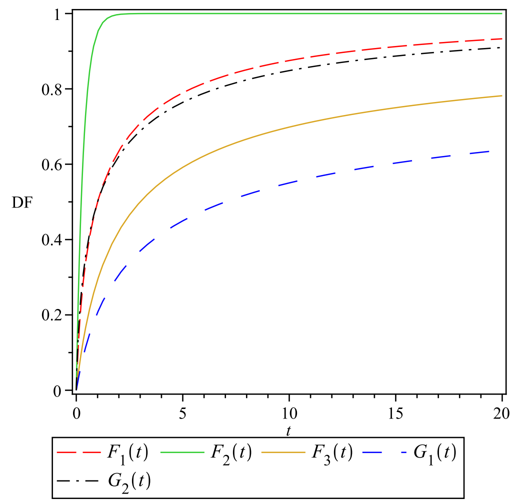

Figure 3: The plot of marginal DFs of main and redundant components.

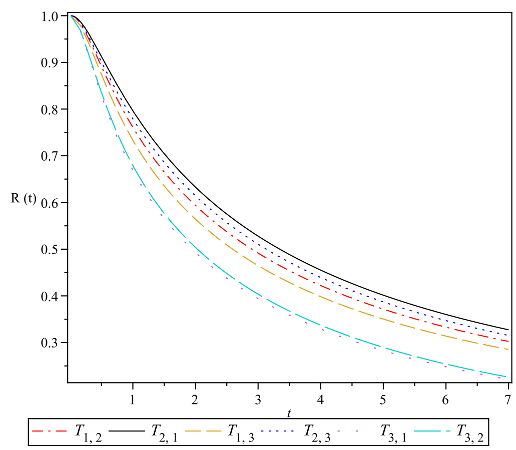

Figure 4: The plot of reliability functions of $T_{i,j}$, $i = 1, 2, 3$, $j = 1, 2$.

Proposition 2.8. *Suppose that $Y_1 \geq_{st} Y_2$ and (Y_1, Y_2) are independent of the main components of a k-out-of-n system. Then $T_{i,j} \geq_{st} T_{j,i}$ if and only if, for all $t \geq 0$,*

$$\beta_i^*(t) \geq \beta_j^*(t), \tag{4}$$

where

$$\beta_i^*(t) = P(\mathcal{B}^{-(i,j)}(t) \geq k-1, X_i > t) - P(\mathcal{B}^{-(i,j)}(t) \geq k-2, X_i > t),$$
$$\beta_j^*(t) = P(\mathcal{B}^{-(i,j)}(t) \geq k-1, X_j > t) - P(\mathcal{B}^{-(i,j)}(t) \geq k-2, X_j > t).$$

Using the simulation method, by considering the MRL criterion we present an example that illustrates the optimal allocation of two spares in a k-out-of-n system.

Example 2.9. Suppose that in a 3-out-of-4 system with two spares, the marginal DFs of components are

$$F_1(t) = 1 - \exp(-t), \; F_2(t) = 1 - \exp(-t^{0.5}),$$

$$F_3(t) = 1 - (1+t)^{-1},\ F_4(t) = 1 - \exp(-t^3),$$
$$G_1(t) = 1 - \exp(-t^{1/5}),\ G_2(t) = 1 - (1+t)^{-1},$$

respectively. In Table 2 we calculated the MRL of $T_{i,j},\ i,j = 1,\ldots,4,\ i \neq j$ at time $x = 0.9395076$ for different copula functions with different parameters. For example Table 2.9 shows that when the joint DF of $(X_1,\ldots,X_4,Y_1,Y_2)$ is given by Frank copula function with parameter $\beta = 1.5$, as

$$\begin{aligned} H(x_1,\ldots,x_4,&y_1,y_2) \\ &= -\frac{1}{\beta}\log(1 + \frac{\prod_{i=1}^{4}(\exp(-\beta F_i(x_i)) - 1)\prod_{j=1}^{2}(\exp(-\beta G_j(y_j)) - 1)}{\exp(-\beta) - 1}), \end{aligned}$$

the MRL of $T_{1,4}$ is maximum with amount of 1.81. This means that if the spares with lifetime Y_1 and Y_2 are allocated to the components with lifetime X_1 and X_4, respectively, then the MRL of system is maximal.

Table 2. The MRLs of $T_{i,j},\ i,j = 1,\ldots,4,\ i \neq j$ in a 3-out-of-4 system at $x = 0.9395076$.

Gumbel copula			Frank copula		
α	Optimal lifetime	MRL	β	Optimal lifetime	MRL
1.5	$T_{1,4}$	7.95	0.5	$T_{1,4}$	1.29
2.0	$T_{1,4}$	14.08	1.0	$T_{1,4}$	1.52
2.5	$T_{1,4}$	13.56	1.5	$T_{1,4}$	1.81

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