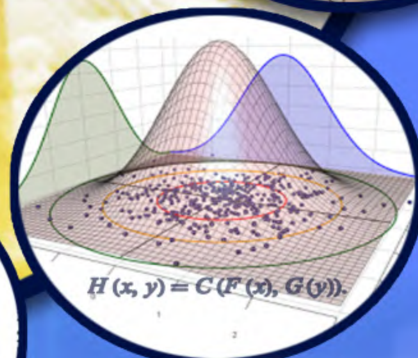

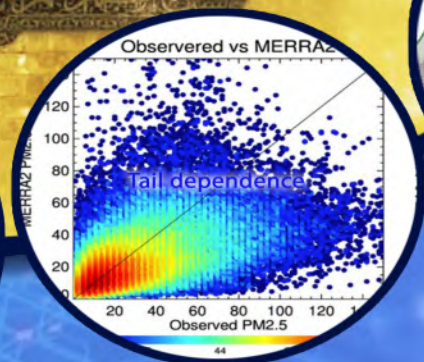

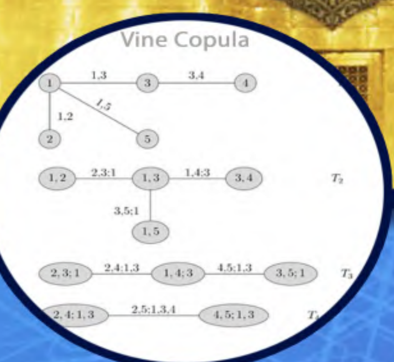

$$W(u,v) = \max(u+v-1,0)$$
$$\Pi(u,v) = uv$$
$$M(u,v) = \min(u,v)$$

هفتمین سمینار
نظریه مفصل و کاربردهای آن

دانشکده علوم ریاضی، دانشگاه فردوسی مشهد، ۱۹ و ۲۰ بهمن‌ماه ۱۴۰۱

The 7th Seminar on Copula Theory and Its Applications

Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, 8 - 9 Feb. 2023

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In the Name of Allah

Proceeding of

The 7th Seminar on

Copula Theory and its Applications

Department of Statistics

and

Ordered Data, Reliability and Dependency Center of Excellence

Ferdowsi University of Mashhad,

Mashhad, Iran

Feb. 8-9, 2023

Disclaimer

This book contains the fulls booklet of the 7th Seminar on "Copula Theory and its Applications". Authors are responsible for the contents and accuracy. Opinions expressed may not necessarily reflect the position of the scientific and organizing committees.

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Preface

The series of biennial workshops on ”Copula Theory” which took place in Ferdowsi University of Mashhad (2011 and 2013), Shahid Bahonar University of Kerman (2015) and Yazd University (2017) with an emphasis on application in engineering sciences, agricultural sciences, actuarial science, finance, reliability, survival analysis, economics and etc. is the result for the decision of the scientific committee of the Ordered and Spatial Data Center of Excellence (OSDCE) of Ferdowsi University of Mashhad (FUM) to organize workshops and seminars every two years. This seminar is sponsored by the department of statistics, OSDCE of FUM, Islamic world Science Citation database (ISC), Iranian Statistical Society and Actuarial Society of Iran to provide suitable facilities for academics to have efficient research cooperation and will be held at Faculty of Mathematical Sciences of FUM at 8 and 9 Feb. 2023. We hope all of the seminar committees provide a suitable satisfactory atmosphere for the participants. After the first call of the seminar, 30 papers were accepted as oral presentations by the referees and scientific committee. The attendants and participants in the seminar are in summary 40 people which are professors, students and researchers of different institutes around Iran. Finally, we would like to extend our sincere gratitude to the Research Council of FUM, the administration of Faculty of Mathematical Sciences, the OSDCE, the Islamic world Science Citation center, the Iranian Statistical Society, Actuarial Society of Iran, the scientific committee, the organizing committee, the referees, and the students and staff of the department of statistics of FUM for their kind cooperation.

Mohammad Amini (Chair)
Feb. 2024

Topics

The aim of the seminar is to provide a forum for presentation and discussion of scientific works covering theories and methods such as:

- Methods of copula construction
- Copula functions and dependence concepts
- Dependence modelling using copula function
- Inference based on copula
- Applications of copula function in spatial, survival, reliability, engineering, hydrological, meteorological, agricultural, finance, economic data and etc.

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The 7th seminar on
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On the copula-based time between events control chart

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Abstract

Time Between Events (TBE) control charts are developed in this article to monitor processes with multiple dependent production lines. An EWMA-type TBE chart has been proposed for this. The copula approach describes production line dependence, while the homogeneous Poisson process lines defects. The proposed methods are evaluated using an average time-to-signal metric.
Keywords: Time Between Events, EWMA chart, Average time to signal, Homogeneous Poisson process

1 Introduction

Control charts for counts is widely used in industry and other hands. In general, researchers have studied the *np* chart for the number of counts, the *p* chart for the proportion of counts, *u* - chart and *c* - chart for nonconformities in process. However, Shewhart control chart is not a good tool to monitore process when the process has less of nonconforming, in this case we may encounter the possibility of a false alarm. To solve like this problem, the time between events (TBE) chart

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is a good tool to monitore the the quantity of conforming items between successive occurrence of nonconforming items. Some examples of TBE data are the number of hours between failures of, for example, valves [8], time between outbreaks of diseases in syndromic surveillance [15], time between communication events in a social network [11], and time between accidents for monitoring occupational safety [14].

Recently, control charts based on TBE became commonly, especially for univariate data [4]. With TBE data, one measures the time elapsed between successive events of interest, e.g., manufacturing defects. These are also known as inter-arrival time data. Within the manufacturing realm, this framework is needed due to increasing process quality, which has resulted in so-called highquality processes. In high-quality processes, the rate of events or defects is very low and hence the number of defects per sample over any reasonable aggregation period is usually zero or very small. Some researchers have extended the univariate TBE control charts into the bivariate or multivariate TBE datasets (see, for example [17, 18]). [9] focused on a production process involving three consecutive sub-processes and regarded the production time of a batch of products on each sub-process. Recently, [16] applied a multivariate cumulative sum TBE chart to monitor the patient's relief time.

Although most of the existing techniques in the literature regarding TBE charts have been proposed for single-unit production processes, there are many real processes consisting of multiple production lines (see for example [19, 7]) or multiple production stages (see for example [5]).

In a process with multiple production lines, in the sense we consider in this study, more than one production line (or machine) are set up in parallel to produce a single product. In such a process, production lines may operate independently or dependently. Reviewing the existing literature on the TBE chart, there is no study discussing the TBE idea for such a process. Accordingly, being motivated by the effectiveness of the TBE chart in monitoring high-quality processes as well as rapidly spreading of manufacturing processes with multiple production lines, the main aim of this study is to develop TBE control charts for monitoring the quality of a manufacturing process with multiple-parallel production lines. Moreover, given that there are likely to be shared resources between production lines of a process (like the same raw material or common power source and operators), it is logical to assume that the quality of different production lines can be dependent. Under this assumption, if one of the production lines or even some of them deviates from its in-control (IC) state, there would be a high chance of deviation in the quality of the other lines. To model such a dependency, copula models have been applied in this study [13, 2]. Copula functions enable us to describe a wide range of dependencies between production lines [1].

Although we defined the problem from an industrial perspective to justify the need for this work and to describe its applicability in practice, the proposed approaches can be applied efficiently in other domains of application like the health monitoring area.

Eventually, the main contribution of the current study to the TBE chart literature is providing an EWMA-type TBE control chart to assess the stability of a multiple lines process that its quality is expressed by the time interval between producing two successive defective items. The proposed techniques have been developed under the assumption that the number of defective times in production lines follows a homogeneous Poisson process. As a result, the time between two successive defectives follows an exponential distribution. The exponential distribution is probably the most popular lifetime model that adequately describes several types of phenomena [3, 12]. A discrete Markov Chain approach is applied to design the EWMA-TBE control chart.

In this work, we use a new method to monitoring processes which has consists of two components using copula function.

2 Motivation

Let us consider a manufacturing process with $v = 2$ parallel production lines, namely line 1 and line 2, operating to produce a single item. Each line produces its own products, from raw materials to ready-to-use items, and there is no direct relationship between their operations. It is further assumed that the number of nonconformities of each line arrives according to homogeneous Poisson processes. Thus, the time Y_i between two successive defectives on production line i $(i = 1, 2)$ follows an Exponential distribution with parameter λ_i, i.e., $Y_i \sim Exp(\lambda_i)$. The probability density function (PDF) and cumulative distribution function (CDF) of the Exponential distribution are as follows:

$$f_{Y_i}(y|\lambda_i) = \frac{1}{\lambda_i} \exp\left(-\frac{y}{\lambda_i}\right) \tag{2.1}$$

$$F_{Y_i}(y|\lambda_i) = 1 - \exp\left(-\frac{y}{\lambda_i}\right), \lambda_i > 0, y > 0, \quad \text{for} \quad i = 1, 2. \tag{2.2}$$

In order to monitor the quality of such a process with two parallel production lines using the TBE approach, we propose applying the variable $T = \min\{Y_1, Y_2\}$ as the monitoring statistic). Then, the monitoring procedure would be carried out by plotting the observed values of this statistic against proper control limits. This study introduced monitoring techniques based on the idea of EWMA control charts. Furthermore, although no direct relationship is supposed to exist between the production lines, some shared resources like raw material, power source, operator, and environmental conditions can make their qualities dependent. To handle this phenomenon, it is considered that the TBEs Y_1 and Y_2 have a joint distribution function H which is fully known once a proper copula model C is determined. This study chooses three copula models from the well-known Archimedean family of copulas, including the Clayton copula, the Gumbel copula, and the Frank copula. Some preliminaries regarding the copula models are provided in the next section.

Eventually, it is worth mentioning to the developed methods would be suitable to check the stability of the process in Phase II. This means that all the process parameters are assumed to be known in advance.

3 Copula Theory

Copula modelling has become an increasingly widespread tool to model dependencies in various domains of applications. Copula helps us to extract the dependence structure from the joint distribution function of a set of variables and, at the same time, split the dependence structure from the univariate marginals. Using the Sklar's theorem, the joint distribution $H(y_1, y_2)$ of the variables Y_1 and Y_2 with marginal CDFs F_{Y_1} and F_{Y_2} can be derived as:

$$H(y_1, y_2|\lambda_1, \lambda_2, C) = C\left(F_1(y_1|\lambda_1), F_2(y_2|\lambda_2)\right), \tag{3.1}$$

once the parametric form of the copula model C is determined. Sklar's theorem also states that when marginal distributions are continuous, which is the case of our study, a unique copula C exists such that (3.1) holds. Before going through the copula models, let us define the dependence measure Kendall's tau τ. There are various kinds of copula models in the literature. This paper concentrates on the three models from the bivariate Archimedean family of copulas that listed in Table 1.

Table 1: Proposed copula models and their measures of dependence

Copula	Copula function	Kendall's tau	Space of θ	λ_L	λ_U
Clayton	$\left[\max\left(u^{-\theta} + v^{-\theta} - 1, 0\right)\right]^{-1/\theta}$	$\theta/(\theta+2)$	$[-1, \infty)\backslash\{0\}$	$2^{\frac{-1}{\theta}}$	0
Frank	$-\frac{1}{\theta}\log\left[1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right]$	$1 + 4\left(\frac{1}{\theta} \int_0^\theta \frac{\mathrm{t}}{\mathrm{e}^{\mathrm{t}}-1} \mathrm{dt} - 1\right)/\theta$	$(-\infty, \infty)\backslash\{0\}$	0	0

4 EWMA control chart

Let $\mathbf{Y}_i = (Y_{1i}, Y_{2i})$ for $i = 1, 2, 3, ...$ be a bivariate vector representing the times until producing defective items by the production lines 1 and 2. For example, if the first line produces its first defective item at time X_1 and produces the second defective item at time X_2, then $Y_{11} = X_1$ and $Y_{12} = X_2 - X_1$. Let consider again the pair (Y_{1i}, Y_{2i}) has the joint distribution H with exponential marginals F_{Y_1} and F_{Y_2} and the known copula C. Here, we are interested in monitoring the process to detect the possible deviations in vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ from its IC value $\boldsymbol{\lambda}_0 = (\lambda_{10}, \lambda_{20})$ to the out-of-control (OC) state $\boldsymbol{\lambda}_1 = (\lambda_{11}, \lambda_{21})$ as quickly as possible. Defining the monitoring statistic $T_i = \min(Y_{1i}, Y_{2i})$, the plotting statistic of the proposed EWMA chart is defined as:

$$Z_i = rT_i + (1-r)Z_{i-1}, \quad i = 1, 2, ..., \tag{4.1}$$

where r is the smoothing parameter such that $0 < r \leq 1$. The initial value of the EWMA statistic can be set as $Z_0 = E[T]$. It is not possible to derive the CDF of Z_i in a closed form, so it is not possible to calculate the control limits of the EWMA-TBE chart in the same way in Shewhart-type. The next subsection provides a procedure to derive the control limits in an optimal manner. However, the control limits of the EWMA-TBE chart should be calculated such that the chart meets a target ATS_0.

4.1 ATS computations

Average run length (ARL) is one of the most popular measures to quantify the performance of a control chart. ARL is the expected value of Run Length (RL) which is defined as the number of samples taken before an OC signal (the monitoring statistic falls beyond the control limits). When the successive monitoring statistics are independent, the Run Length distribution is Geometric with the probability of success equal to p which means that $ARL = \frac{1}{p}$. While the process is in IC (OC) state, the probability p is equal to the probability of type I error α (1 - the probability of type II error β). These lead to the IC and OC ARLs as $ARL_0 = \frac{1}{\alpha}$ and $ARL_1 = \frac{1}{1-\beta}$, respectively. For the proposed Shewhart-type TBE chart, these probabilities can be calculated as:

$$\alpha = F_T(LCL|\boldsymbol{\lambda}_0) + 1 - F_T(UCL|\boldsymbol{\lambda}_0) = \alpha_L + \alpha_U \tag{4.2}$$

$$\beta = F_T(UCL|\boldsymbol{\lambda}_1) - F_T(LCL|\boldsymbol{\lambda}_1), \tag{4.3}$$

where F_T is given by

$$\begin{aligned} F_T(t|\boldsymbol{\lambda}) &= Pr(T \leq t|\boldsymbol{\lambda}) \\ &= 1 - Pr(\min(Y_1, Y_2) \geq t|\boldsymbol{\lambda}) \\ &= 1 - (1 - P(Y_1 \leq t) - P(Y_2 \leq t) + P(Y_1 \leq t, Y_2 \leq t)) \\ &= F_{Y_1}(t|\lambda_1) + F_{Y_2}(t|\lambda_2) - C\left(F_{Y_1}(t|\lambda_1), F_{Y_2}(t|\lambda_2)\right). \end{aligned} \tag{4.4}$$

Employing equation (4.4) for the mentioned copula models, distribution function F_T can be obtained. The suitable performance metric for TBE control charts is ATS which is defined as the average expected time from the start of the process until the chart signals [10]. ATS can also be categorized into two types, IC and OC ATS denoted by ATS_0 and ATS_1, respectively. For the proposed Shewhart-type TBE, the time to signal TS can be calculated as:

$$TS = \sum_{i=1}^{RL} T_i. \tag{4.5}$$

If RL and T are independent, then we have:

$$ATS = E(TS) = E(T) \times E(RL) = \mu_T \times ARL, \tag{4.6}$$

where $\mu_T = E(T)$. Therefore, ATS_0 and ATS_1 can be computed as:

$$ATS_0 = \frac{1}{\alpha} \int_0^\infty \bar{F}_T(t|\boldsymbol{\lambda}_0)dt \tag{4.7}$$

$$ATS_1 = \frac{1}{1-\beta} \int_0^\infty \bar{F}_T(t|\boldsymbol{\lambda}_1)dt, \tag{4.8}$$

where α and β are given in (4.2) and (4.3) and $\bar{F}_T = 1 - F_T$ where F_T is presented in (4.4).

The ATS of the EWMA-TBE chart can not be calculated straightforwardly. So, we use a discrete Markov Chain approach proposed by [6] to evaluate the ATS of the proposed EWMA-TBE control chart. In what follows, we propose a constrained optimization problem to calculate the unknown parameters (LCL, UCL, r) such that the control chart meets some desired statistical properties. To calculate the chart parameters so that it can detect a specific shift as fast as possible while being ATS-unbiased and meeting ATS_0, we propose to solve the following optimization problem:

$$\begin{aligned} &Minimize \quad ATS(LCL, UCL, r|\boldsymbol{\lambda}_1) \\ &Subject \quad to \quad ATS(LCL, UCL, r|\boldsymbol{\lambda}_0) = ATS_0 \\ &\qquad ATS(LCL, UCL, r|\boldsymbol{\lambda}_1) \leq ATS_0 \\ &\quad r \in (0,1], \quad 0 < LCL < UCL. \end{aligned} \tag{4.9}$$

5 Simulation Study

In order to assess the performance of the proposed TBE control charts and investigate their behaviors with respect to various process parameters, this section aims to conduct a numerical study based on the ATS metric. First of all, let us set $ATS_0 = 370$. In addition, suppose that Y_1 and Y_2 are jointly distributed with the known copula model C and exponential marginals given in (2.1) with the IC parameter $\boldsymbol{\lambda}_0 = (\lambda_{10}, \lambda_{20}) = (2, 3)$. It is also assumed that τ takes its values from the set $\{0.3, 0.8\}$ to cover both the moderate and strong dependences when the TBEs Y_1 and Y_2 are positively or negatively dependent.

To calculate the ATS_1 values, the OC vector of the parameters is also considered as $\boldsymbol{\lambda}_1 = (\lambda_{11}, \lambda_{21}) = (\delta_1\lambda_{10}, \delta_2\lambda_{20})$ where $\delta_1 = 0.25, 0.50, 1.50, 2.00$ and $\delta_2 = 0.25, 0.75, 1.25, 2.00$.

This setting helps us to assess the charts' ability to detect shifts in λ_1 and λ_1 when they deviate from the IC state in four directions: increasing-increasing, increasing-decreasing, decreasing-increasing, and decreasing-decreasing.

Table 2 shows the results of the numerical analysis of the EWMA-TBE chart. The optimal charts' parameters r, LCL, and UCL as well the ATS_1 value regarding these parameters are the outputs of the optimization problem (4.9). From this table, the following results are drawn:

- *Biasedness*: The EWMA-TBE chart is ATS-unbiased across all shift combinations (δ_1, δ_2), i.e., we have always $ATS_0 > ATS_1$.

- *Sensitivity with respect to* τ, λ_{10}, *and* λ_{20}: The table 2 shows that any changes in these parameters notably affect the chart's ability to detect OC states. For example, when $\delta_1 = 1.25, \delta_2 = 0.75$, the ATS_1 values of chart based on Frank copula are $239.80, 297.77, 352.56$, and 254.71 when $\tau = -0.8, -0.3, 0.3, 0.8$, respectively.

6 Conclusions

This article proposed an EWMA-type control chart for monitoring the quality of a process with multiple production lines using the time between events approach. The proposed methods' performance is evaluated based on the ATS metric. Numerical computations have been done to assess the performance of the monitoring procedures with respect to different process parameters. In brief, the results of the numerical study showed that the process parameters and the dependence parameter remarkably affect the OC performance of both charts and the proposed EWMA-TBE chart is ATS-unbiased.

Future works can be considered to investigate the performance of the proposed control charts in the presence of other types of defective processes like the non-homogeneous Poisson model and can extend the proposed approaches in order to monitor the TBE dataset when there are more than two production lines.

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Table 2: The optimal EWMA-TBE chart's parameters (LCL, UCL, r) and ATS_1 values of the Clayton copula when $\boldsymbol{\lambda}_0 = (2, 3)$.

(δ_1, δ_2)	$\tau = -0.8$				$\tau = -0.3$			
	LCL^*	UCL^*	r^*	ATS	LCL^*	UCL^*	r^*	ATS
Clayton								
(0.25,0.25)	0.20815	3.58219	0.47941	5.59	0.26839	3.71165	0.39979	6.33
(0.50,0.25)	0.35828	2.03522	0.24774	8.91	0.30459	4.41992	0.36469	11.10
(0.50,0.75)	0.45828	2.47617	0.16558	28.66	0.39315	3.34816	0.26348	35.12
(1.50,0.75)	0.38122	0.97318	0.04911	218.04	0.52068	1.22286	0.03900	226.75
(0.50,1.25)	0.58361	1.29663	0.07511	51.83	0.63859	1.99065	0.09735	60.53
(2.00,1.25)	0.26299	1.46524	0.20839	51.20	0.49884	1.44187	0.07721	55.64
(2.00,2.00)	0.37192	1.23719	0.12940	28.81	0.39524	1.66608	0.12401	40.56
Frank								
(0.25,0.25)	0.29697	1.74622	0.26657	1.36	0.26850	3.23277	0.33858	2.16
(0.50,0.25)	0.31450	1.56144	0.24483	2.56	0.35812	2.77649	0.24193	4.24
(0.50,0.75)	0.45951	1.32306	0.11537	9.37	0.46959	2.39967	0.15600	15.49
(1.25,0.75)	0.70824	0.88971	0.01000	239.80	0.66754	3.19835	0.06240	297.37
(0.50,1.25)	0.51387	1.07198	0.07607	18.57	0.58945	1.97494	0.09097	27.24
(2.00,1.25)	0.09243	1.75714	0.60187	9.54	0.28158	1.76409	0.14543	30.52
(2.00,2.00)	0.01338	1.91626	0.79589	5.35	0.38033	1.56823	0.10251	19.23
(δ_1, δ_2)	$\tau = 0.3$				$\tau = 0.8$			
Clayton								
(0.25,0.25)	0.39092	4.71960	0.35326	3.07	0.54097	5.52803	0.35174	3.72
(0.50,0.25)	0.50150	3.63087	0.25261	6.26	0.65621	5.03795	0.28377	7.73
(0.50,0.75)	0.83868	2.74244	0.10439	18.39	1.02182	3.79977	0.13741	17.06
(1.25,0.75)	1.28014	2.29049	0.01950	338.50	0.78957	2.25157	0.03522	285.10
(0.50,1.25)	0.91214	2.55275	0.08274	25.11	1.01855	3.81607	0.13831	17.57
(2.00,1.25)	0.56778	2.36864	0.11755	36.47	0.64140	3.25509	0.15490	37.87
(2.00,2.00)	0.44377	2.67362	0.16748	25.62	0.46970	3.73081	0.22513	29.11
Frank								
(0.25,0.25)	0.36911	4.80471	0.34942	3.45	0.49569	6.39568	0.37606	4.06
(0.50,0.25)	0.56777	3.84393	0.21339	6.81	0.84170	4.64207	0.19685	8.39
(0.50,0.75)	0.75107	3.43396	0.13487	20.21	1.06958	3.80232	0.12341	17.74
(1.25,0.75)	0.98322	3.43274	0.07001	352.56	0.89877	2.34357	0.03701	254.71
(0.50,1.25)	0.67412	4.08837	0.16666	28.01	1.05846	3.85811	0.12683	17.97
(2.00,1.25)	0.53647	2.46707	0.11517	38.77	0.60467	3.44515	0.16093	39.26
(2.00,2.00)	0.36468	2.94430	0.18837	27.57	0.46937	3.85522	0.21668	31.53

Investigating the Effect of Dependence of Fading Coefficients and its Modeling with Copula Theory in Non-Orthogonal Multiple Access (NOMA) Channels with Physical Layer Security

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Abstract

In wireless communication, in most mathematical modeling, it is assumed that the fading coefficients are independent of each other, if physically, there is a correlation between them. In this paper, the non-orthogonal multiple access (NOMA) Downlink with physical layer security and dependent fading coefficients is investigated.The average secrecy rate (ASR) for the NOMA channel in the presence of an eavesdropper has been investigated by modeling the dependence of extinction coefficients by copula functions. With mathematical calculations and numerical results, we compared the effect of correlation in the studied fading coefficients and independent fading coefficients to find out whether this modeling is useful or harmful.

Keywords: Non-orthogonal multiple access (NOMA), Physical layer security, Copula functions, Average secrecy rate

1 Introduction

The Non-orthogonal multiple access scheme is known as one of the effective techniques in multiple access, which has the ability to improve spectral efficiency and fairness of users[1, 6, 2]. In wireless communication of the first to fourth generation, the idea of orthogonal use of available channel

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resources was proposed in order to prevent channel interference between each user, Therefore, the number of users who could use the channel's resources was limited. In order to separate the overlapping messages of different users, the technique of Successive Interference Cancellation (SIC) in the receivers is used[3]. Unlike OMA schemes, in NOMA, the superimposed messages of all multiple users are sent simultaneously on the entire channel, so there is a risk that an eavesdropper can hear these messages. Therefore, in NOMA, there is a necessity to secure confidential messages in case of illegal use [4]. A joint multivariate distribution function is used to investigate the performance of NOMA channels with dependent fading coefficients, so having an efficient and appropriate mathematical tool can help to investigate the dependence in NOMA channels. It is suggested to apply and use copula functions as an effective method to express the dependence between variables. Copula expresses joint distributions by applying marginal distribution functions, and these joint distributions, which have different types. Copula functions are used in many sciences including statistics, machine learning, image processing and many applications in engineering.[5, 20]. The physical layer security has been studied by many researchers for non-orthogonal multiple access channels [13, 8]

Our work. In this paper, we investigate the downlink NOMA with two legitimate users (one strong user and the other weak user) in the presence of an eavesdropper. It is assumed that there is a dependence between the fading coefficients, and we model this dependence using copula functions. The joint probability density function between fading coefficients is obtained with the help of copula functions, and the average secrecy rates of each user is calculated. Then the effect of dependence on the performance of each user is compared to the situation where the fading coefficients are independent.

2 A Brief Review of Copula Theory

In order to model the dependence between channel fading coefficients, the copula function is used. Copula is a multivariate cumulative distribution function so that the distribution of marginal probabilities of each variable in the interval $[0,1]$ has a uniform distribution [5]. It is necessary to express the probability density function (PDF) of channel fading coefficients with the help of copula functions in order to calculate the average secrecy rate in NOMA systems. Suppose that $\mathcal{S}=(X_m,X_n)$ is a vector of two random variables with Cumulative Distribution Function (CDF) as $F(x_m,x_n)$ and marginal CDFs $F(x_m)$ and $F(x_n)$ respectively.

A copula is a function $C:[0,1]^2\rightarrow[0,1]$ which satisfies following properties:

For each u and v in the interval $[0,1]$ we have:

$$c\,(u,0)=c\,(0,v)=0,\qquad c\,(u,1)=u\qquad\text{and}\qquad c\,(1,v)=v \tag{2.1}$$

The 2-increasing property, for every u_1,u_2,v_1,v_2 in the interval $[0,1]$ such that $u_1\leq u_2$ and $v_1\leq v_2$:

$$C\,(u_2,v_2)-C\,(u_1,v_2)-C\,(u_2,v_1)+C(u_1,v_1)\geq 0 \tag{2.2}$$

Theorem 2.1. *(Sklar's theorem): This theorem states that any multivariate CDF of a random variable can be expressed in terms of marginal functions. Let $F(x_m, x_n)$ as CDF of random variables with margins $F(x_i)$ for $i = m, n$. Then there exists a copula C such that for all x_m, x_n in $\bar{R}$ [4].*

$$F\left(x_m, x_n\right) = C(F\left(x_m\right), F\left(x_n\right)) \tag{2.3}$$

Then, according to the Sklar's theorem, the joint PDF is obtained for the marginal functions $F\left(x_m\right)$ and $F\left(x_n\right)$ respectively.

$$f\left(x_m, x_n\right) = f\left(x_m\right) f\left(x_n\right) c(F\left(x_m\right), F\left(x_n\right)) \tag{2.4}$$

Where $c\left(F\left(x_m\right), F\left(x_n\right)\right) = \frac{\partial^2 C(F(x_m), F(x_n))}{\partial F(x_m) . \partial F(x_n)}$ is the Copula density function. Also $f\left(x_m\right)$ and $f\left(x_n\right)$ are marginal PDFs, respectively. There are many different copulas that can be used.

In this paper, we use FGM copula to analyze the performance criteria of the proposed system from an empirical point of view, that the FGM copulas are the simplest mode for calculating the joint PDFs [5] and consider negative and positive correlations and independence situation. FGM copulas are defined as follows:

$$C\left(u, v\right) = uv(1 + \theta\left(1 - u\right)\left(1 - v\right)) \tag{2.5}$$

where $\theta \epsilon \left[-1, 1\right]$ is defined as the dependence parameter and $u = F_{X_m}(x_m)$ and $v = F_{X_n}(x_n)$. Negative and positive values of θ indicate negative and positive dependence, respectively, and for zero value, we have independence.

3 Secrecy Rate Region of NOMA:

Theorem 3.1. *The average secrecy rates (ASC) for users with dependent fading coefficients are obtained as follows:*

$$R_{n,ave}^{Sec} \leq E_{\gamma_m, \gamma_e}[\log\left(1 + \alpha_m \gamma_m\right) - \log\left(1 + \alpha_m \gamma_e\right)]^+ \tag{3.1}$$

$$R_{n,ave}^{Sec} \leq E_{\gamma_n, \gamma_e}[\log\left(1 + \frac{\alpha_n \gamma_n}{\alpha_m \gamma_n + 1}\right) - \log\left(1 + \frac{\alpha_n \gamma_e}{\alpha_m \gamma_e + 1}\right)]^+ \tag{3.2}$$

where α_i for $i = m, n$ expresses power allocation factors for each user, where $\alpha_m + \alpha_n = 1$ and $0 \leq \alpha_m \leq \alpha_n \leq 1$. In relations(3.1) and (3.2), the value of γ_i is equal to $\gamma_i = \frac{P|h_i|^2}{N_i}$ for $i = m, n, e$.

Lemma 3.2. *The joint PDF of* γ_i *and* γ_j $(f(\gamma_i,\gamma_j))$ *based on Farlie-Gumbel-Morgenstern (FGM) Copula is determined as:*

$$f(\gamma_i,\gamma_j)=\frac{e^{\frac{-\gamma_i}{\bar{\gamma}_i}-\frac{\gamma_j}{\bar{\gamma}_j}}}{\bar{\gamma}_i\bar{\gamma}_j}[1+\theta\left(1-2e^{\frac{-\gamma_i}{\bar{\gamma}_i}}\right)\left(1-2e^{\frac{-\gamma_j}{\bar{\gamma}_j}}\right)] \tag{3.3}$$

that the marginal probability density functions in relation (3.4) are defined as follows:

$$f_{\gamma_i}(\gamma_i)=\frac{1}{\bar{\gamma}_i}e^{\frac{-\gamma_i}{\bar{\gamma}_i}} \quad i=m,n,e \quad \gamma_i>0 \quad \bar{\gamma}_i=\frac{1}{2\sigma_{h_i}^2} \tag{3.4}$$

By simplifying and mathematical calculations, the ASC of the user m is calculated as follows:

$$\begin{aligned}R^{sec}_{m,ave}\leq[&-e^{\frac{1}{\alpha_m\bar{\gamma}_m}}Ei\left(-\frac{1}{\alpha_m\bar{\gamma}_m}\right)\\&+e^{\left(\frac{\bar{\gamma}_e+\bar{\gamma}_m}{\alpha_m\bar{\gamma}_e\bar{\gamma}_m}\right)}Ei\left(-\frac{\bar{\gamma}_e+\bar{\gamma}_m}{\alpha_m\bar{\gamma}_e\bar{\gamma}_m}\right)\theta(-e^{\left(\frac{\bar{\gamma}_e+\bar{\gamma}_m}{\alpha_m\bar{\gamma}_e\bar{\gamma}_m}\right)}Ei\left(-\frac{\bar{\gamma}_e+\bar{\gamma}_m}{\alpha_m\bar{\gamma}_e\bar{\gamma}_m}\right)\\&+e^{\left(\frac{2\bar{\gamma}_e+\bar{\gamma}_m}{\alpha_m\bar{\gamma}_e\bar{\gamma}_m}\right)}Ei\left(-\frac{2\bar{\gamma}_e+\bar{\gamma}_m}{\alpha_m\bar{\gamma}_e\bar{\gamma}_m}\right)+e^{\left(\frac{\bar{\gamma}_e+2\bar{\gamma}_m}{\alpha_m\bar{\gamma}_e\bar{\gamma}_m}\right)}Ei\left(-\frac{\bar{\gamma}_e+2\bar{\gamma}_m}{\alpha_m\bar{\gamma}_e\bar{\gamma}_m}\right)\\&-e^{\left(\frac{2\bar{\gamma}_e+2\bar{\gamma}_m}{\alpha_m\bar{\gamma}_e\bar{\gamma}_m}\right)}Ei\left(-\frac{2\bar{\gamma}_e+2\bar{\gamma}_m}{\alpha_m\bar{\gamma}_e\bar{\gamma}_m}\right))]^+\end{aligned} \tag{3.5}$$

The average secrecy rate of the user n is also calculated as follows:

$$\begin{aligned}R^{Sec}_{n,ave}\leq[&e^{\frac{1}{\alpha_m\bar{\gamma}_n}}Ei\left(-\frac{1}{\alpha_m\bar{\gamma}_n}\right)-e^{\frac{1}{\bar{\gamma}_n}}Ei\left(-\frac{1}{\bar{\gamma}_n}\right)\\&-e^{\left(\frac{\bar{\gamma}_e+\bar{\gamma}_n}{\alpha_m\bar{\gamma}_e\bar{\gamma}_n}\right)}Ei\left(-\frac{\bar{\gamma}_e+\bar{\gamma}_n}{\alpha_m\bar{\gamma}_e\bar{\gamma}_n}\right)+e^{\left(\frac{\bar{\gamma}_e+\bar{\gamma}_n}{\bar{\gamma}_e\bar{\gamma}_n}\right)}Ei\left(-\frac{\bar{\gamma}_e+\bar{\gamma}_n}{\bar{\gamma}_e\bar{\gamma}_n}\right)\\&+\theta(e^{\left(\frac{\bar{\gamma}_e+\bar{\gamma}_n}{\alpha_m\bar{\gamma}_e\bar{\gamma}_n}\right)}Ei\left(-\frac{\bar{\gamma}_e+\bar{\gamma}_n}{\alpha_m\bar{\gamma}_e\bar{\gamma}_n}\right)-e^{\left(\frac{2\bar{\gamma}_e+\bar{\gamma}_n}{\alpha_m\bar{\gamma}_e\bar{\gamma}_n}\right)}Ei\left(-\frac{2\bar{\gamma}_e+\bar{\gamma}_n}{\alpha_m\bar{\gamma}_e\bar{\gamma}_n}\right)\\&-e^{\left(\frac{\bar{\gamma}_e+2\bar{\gamma}_n}{\alpha_m\bar{\gamma}_e\bar{\gamma}_n}\right)}Ei\left(-\frac{\bar{\gamma}_e+2\bar{\gamma}_n}{\alpha_m\bar{\gamma}_e\bar{\gamma}_n}\right)+e^{\left(\frac{2\bar{\gamma}_e+2\bar{\gamma}_n}{\alpha_m\bar{\gamma}_e\bar{\gamma}_n}\right)}Ei\left(-\frac{2\bar{\gamma}_e+2\bar{\gamma}_n}{\alpha_m\bar{\gamma}_e\bar{\gamma}_n}\right)\\&-e^{\left(\frac{\bar{\gamma}_e+\bar{\gamma}_n}{\bar{\gamma}_e\bar{\gamma}_n}\right)}Ei\left(-\frac{\bar{\gamma}_e+\bar{\gamma}_n}{\bar{\gamma}_e\bar{\gamma}_n}\right)+e^{\left(\frac{2\bar{\gamma}_e+\bar{\gamma}_n}{\bar{\gamma}_e\bar{\gamma}_n}\right)}Ei\left(-\frac{2\bar{\gamma}_e+\bar{\gamma}_n}{\bar{\gamma}_e\bar{\gamma}_n}\right)\\&+e^{\left(\frac{\bar{\gamma}_e+2\bar{\gamma}_n}{\bar{\gamma}_e\bar{\gamma}_n}\right)}Ei\left(-\frac{\bar{\gamma}_e+2\bar{\gamma}_n}{\bar{\gamma}_e\bar{\gamma}_n}\right)-e^{\left(\frac{2\bar{\gamma}_e+2\bar{\gamma}_n}{\bar{\gamma}_e\bar{\gamma}_n}\right)}Ei\left(-\frac{2\bar{\gamma}_e+2\bar{\gamma}_n}{\bar{\gamma}_e\bar{\gamma}_n}\right))]^+\end{aligned} \tag{3.6}$$

4 NUMERICAL RESULS:

In this section, the ASR of each users per fixed eavesdropper channel gain SNR $\bar{\gamma}_e$, increases with the increase of each user's channel gains SNR ($\bar{\gamma}_m,\bar{\gamma}_n$). The ASR of each users for different values of parameter θ is shown in Figer 1 ,the ASR of user m for positive dependence has a better performance than the ASR of users with independent joint probability density function.

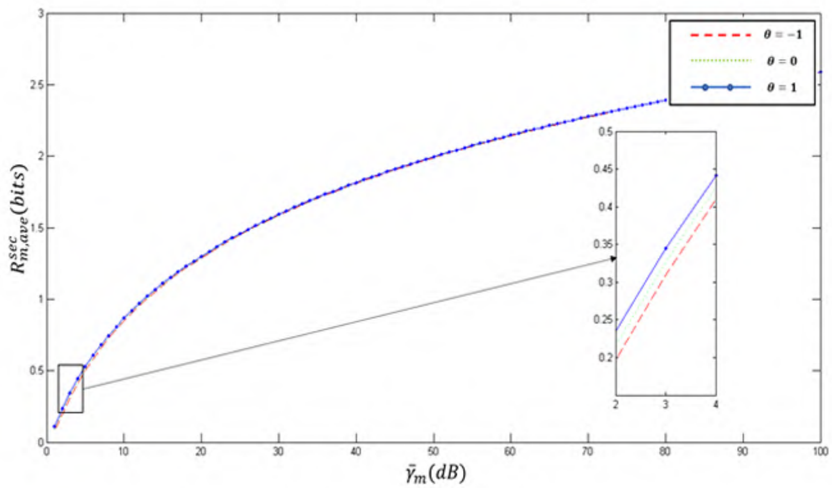

Figure 1: The ASR of user m versus $\bar{\gamma}_m$, for different values of dependence parameter θ.

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Skew-elliptical Distribution of Copula Related Random Variables

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Abstract

In this article, we introduce a generalization of the skew distributions when the random variables are associated to a copula. Considering two Gaussian-copula and t-copula, we present a generalisation of skew-normal as well as skew-t distributions. We investigate the performance of our proposed distributions using a simulation study.
Keywords: Skew-normal distribution , Skew-t distribution, Copula function, Gaussian-Copula function, Student's t-copula function.

1 Introduction

1.1 Copula Theorem

A copula, as explained by Sklar (1959), is a function used to link univariate marginal distributions of random variables into a multivariate distribution. In brief, suppose a 2-dimensional random vector $X = (x_1, x_2)^T$ has its marginal cumulative distribution functions $F_1(x_1)$, $F_2(x_2)$ and probability density functions $f_1(x_1)$, $f_2(x_2)$, Therefore [4]

$$F(x_1, x_2) = C[F_1(x_1), F_2(x_2)] = C(u_1, u_2) \tag{1.1}$$

where C denotes the copula function.

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Gaussian Copula. For a given correlation matrix $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, the Gaussian copula with correlation matrix Σ can be written as

$$C^{Ga}(u_1, u_2) = \Phi_\Sigma(\Phi^{-1}(u_1), \Phi^{-1}(u_2)) \tag{1.2}$$

where Φ_Σ is the joint bivariate distribution function of a Gaussian variable with mean vector zero and correlation matrix Σ and $\Phi^{-1}(.)$ is the CDF of univariate standard normal distribution.

Student's t-copula. For a given correlation matrix $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ the Student's t-copula with correlation matrix Σ and ν degrees of freedom can be written as

$$C^t(u_1, u_2) = T_{\Sigma,\nu}(T_\nu^{-1}(u_1), T_\nu^{-1}(u_2)) \tag{1.3}$$

where $T_{\Sigma,\nu}$ is the joint bivariate distribution function of a Student's t-variable with correlation matrix Σ and ν degrees of freedom and $T_\nu(.)$ is the CDF of univariate Student's-t distribution.

1.2 Skew-Symmetric Distributions

If f_0 is a one-dimensional probability density function symmetric about 0, and G is a one dimensional distribution function such that G_0 exists and is a density symmetric about 0, then $f(z) = 2f_0(z)G(w(z)), z \in R$ is a density function for any odd function $w(.)$ [1].

2 Main results

In this part, we will obtain the distribution of $Z \overset{\mathrm{d}}{=} (X|Y > \mu_y)$ when X, Y are associated to a copula we first state theoretical results, we will examine the obtained results using a simulation study.

Lemma 2.1. *Assume that two continuous random variables X and Y are connected via the copula function $C_{X,Y}$, then the distribution of $Z \overset{\mathrm{d}}{=} (X|Y > \mu_y)$ is given by*

$$f_Z(z) = m_C f_X(z)(1 - D_1 C_{X,Y}), \qquad z \in \mathbb{R}, \tag{2.1}$$

where $D_1 C_{X,Y} = \frac{\partial C_{X,Y}}{\partial F_X(z)}$ if exist otherwise 0 and $m_C = (\int_C f_Y(y)dy)^{-1}$ [2, 3].

Theorem 2.2. *Suppose that variables X and Y with distribution functions $F_X(x)$ and $F_Y(y)$ be connected to each other through gaussian-copula, then the distribution of $Z \overset{\mathrm{d}}{=} (X|Y > \mu_y)$ is given by*

$$f_Z(z) = m_C f_X(z)\Phi(\frac{\rho\Phi^{-1}(F_Z(z)) - \Phi^{-1}(F_Y(\mu_y))}{\sqrt{1-\rho^2}}), \qquad z \in \mathbb{R}, \tag{2.2}$$

Proof. It is easily proved using the above lemma. $\square$

Corollary 1: Suppose X has t-distribution with ν degrees of freedom and Y has a distribution function $F_Y(y)$ with mean μ_y are related by the gaussian-copula with correlation ρ, then the distribution of $Z \stackrel{\mathrm{d}}{=} (X|Y > \mu_y)$ is

$$f_Z(z) = m_C t_1(z;\nu)\Phi(\frac{\rho\Phi^{-1}(T_\nu(z)) - \Phi^{-1}(F_Y(\mu_y))}{\sqrt{1-\rho^2}}) \tag{2.3}$$

Proof. Using the stated theorem we have

$$\begin{aligned} f_Z(z) &= m_C\big[t_1(z;\nu) - \frac{\partial C^{Ga}_{X,Y}}{\partial T_\nu(z)}\frac{\partial T_\nu(z)}{\partial z}\big] \\ &= m_C\big[t_1(z;\nu) - t_1(z;\nu)\frac{\partial C^{Ga}_{X,Y}}{\partial T_\nu(z)}\big] \\ &= m_C t_1(z;\nu)\Phi(\frac{\rho\Phi^{-1}(T_\nu(z)) - \Phi^{-1}(F_Y(\mu_y))}{\sqrt{1-\rho^2}}) \end{aligned}$$

which is 2.3. □

Theorem 2.3. *Suppose that variables X and Y with distribution functions $F_X(x)$ and $F_Y(y)$ be connected to each other through t-copula, then the distribution of $Z \stackrel{\mathrm{d}}{=} (X|Y > \mu_y)$ is given by*

$$f_Z(z) = m_C f_X(z)T(\frac{\rho T^{-1}(F_X(z)) - T^{-1}(F_Y(\mu_y))}{\sqrt{1-\rho^2}}(\frac{\nu+1}{\nu + T^{-1}(F_X(z))^2})^{\frac{1}{2}};\nu+1), \tag{2.4}$$

Proof. It is easily proved □

Corollary 2: Suppose X has normal distribution and Y has a distribution function $F_Y(y)$ with mean μ_y are related by the t-copula with correlation ρ, then the distribution of $Z \stackrel{\mathrm{d}}{=} (X|Y > \mu_y)$ is

$$f_Z(z) = m_C\phi(z)T(\frac{\rho T^{-1}(\Phi(z)) - T^{-1}(F_Y(\mu_y))}{\sqrt{1-\rho^2}}(\frac{\nu+1}{\nu + T^{-1}(\Phi(z))^2})^{\frac{1}{2}};\nu+1) \tag{2.5}$$

where $m_{\mu_y} = (\int_{\mu_y} f_Y(y)dy)^{-1}$.
The proof is similar to the proof of Corollary 1 and is omitted.

2.1 Simulation Results

To check the results presented in Corollary 1, we generated 1000 random pairs of (X_i, Y_i), $i = 1, 2, ..., 1000$, using Monte Carlo simulation so that X has a t distribution with $\nu = 12$ and Y has an exponential distribution with $\lambda = 20$ and they are related to each other through the Gaussian-copula with correlations $\rho = 0.5$ and $\rho = 0.9$. We then repeated this procedure 5000 times. Tables 1 summarized the AIC and BIC of our proposed skew-copula distribution against the skew-t and skew-normal ones. As seen from this table, skew-copula outperforms skew-t and skew-normal, see also figure 1.

Table 1: AIC and BIC of skew-t and skew-normal and skew-copula.

Estimation	skew-t	skew-normal	skew-copula
$AIC_{\rho=0.5}$	1062	1063	1061
$BIC_{\rho=0.5}$	1076	1077	1074
$AIC_{\rho=0.8}$	840	838	838
$BIC_{\rho=0.8}$	853	853	852

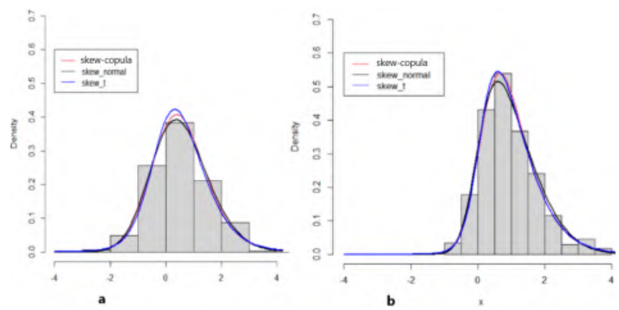

Figure 1: Performance of skew-t and skew-normal and skew-copula, a) $\rho = 0.3$, b) $\rho = 0.8$.

Next, to examine the equation stated in Corollary 2, we generated 1000 random pairs of (X_i, Y_i), $i = 1, 2, ..., 1000$, so that X has a normal distribution and Y has an exponential distribution with $\lambda = 10$ and they are related to each other through the t-copula with correlations $\rho = 0.3$ and $\rho = 0.8$. We then repeated this procedure 5000 times. Tables 2 summarized the AIC and BIC of our proposed skew-copula distribution against the skew-t and skew-normal ones. As you can see in Table 2 and figure 2 skew-copula performs better according to the two criteria AIC and BIC.

Table 2: AIC and BIC of skew-t and skew-normal and skew-copula.

Estimation	skew-t	skew-normal	skew-copula
$AIC_{\rho=0.3}$	1038	1035	1034
$BIC_{\rho=0.3}$	1050	1048	1045
$AIC_{\rho=0.8}$	847	846	845
$BIC_{\rho=0.8}$	858	857	857

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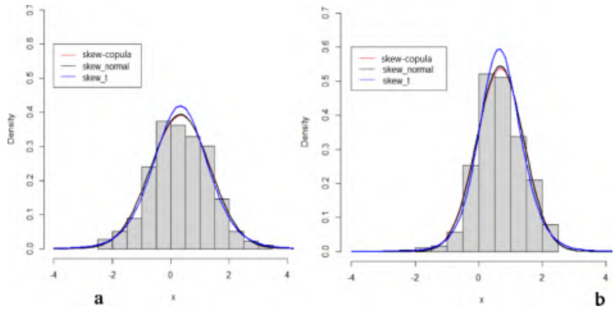

Figure 2: Performance of skew-t and skew-normal and skew-copula, a) $\rho = 0.3$, b) $\rho = 0.8$.

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Stochastic comparisons of extreme order statistics from the generalized Gompertz distribution

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Abstract

Stochastic comparison on order statistics from heterogeneous-dependent observations has been paid lots of attention recently. This paper devotes to investigating the ordering properties of order statistics from dependent observations. In the presence of the Archimedean copula or survival copula for the random variables of samples having generalized Gompertz distribution, we obtain the usual stochastic order of the sample extremes. In addition, some examples illustrating the main results are presented as well.
Keywords: Generalized Gompertz distribution, Majorization, Usual stochastic order, extreme order statistics, Archimedean copula.

1 Introduction

Let $X_{1:n} \leq \ldots \leq X_{n:n}$ denote the order statistics arising from random variables $X_1, \ldots, X_n$. Order statistics play a prominent rule in the reliability theory, life testing, operations research and other related areas. In reliability theory, the kth order statistic coresponds to the lifetime of a $(n-k+1)$-out-of-n system. In particular, $X_{1:n}$ and $X_{n:n}$ correspond to the lifetimes of series and parallel systems, respectively. In recent years, stochastic ordering relations between extreme order statistics from parametric families of distributions have been studied extensively by many researchers. The Gompertz distribution is one of classical mathematical models that represent survival function

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based on laws of mortality. This distribution plays an important role in modeling human mortality and fitting actuarial tables. [3] proposed the exponentiated Gopertz distribution, and referred to it as the generalized Gompertz (GG) distribution. It has a bathtub shaped failure function and can be used to provide a good fit for the real data than well-known distributions. Further, this distribution generalizes some well-known distributions and can be used to model phenomena which are common in reliability and biological studies. The non-negative random variable X is said to have a generalized Gompertz distribution (GGD) with three parameters λ, α, θ if its cumulative distribution function is given by the following form

$$F(x) = [1 - \exp\{-\frac{\lambda}{\alpha}(e^{\alpha x} - 1)\}]^{\theta}, \quad \lambda, \theta, \alpha > 0, x \geqslant 0. \tag{1.1}$$

The parameter θ is a shape parameter. The generalized Gompertz distribution with parameters λ, α and θ will be denoted by $GGD(\lambda, \alpha, \theta)$. The first advantage of GGD is that it has the closed form of its the cumulative distribution function.

In the presence of the Archimedean copula, section 3 studies stochastic comparison of series or parallel dependent systems in terms of the usual stochastic order. To continue our discussion, we need definitions of some stochastic orders and the concept of majorization which is given in Section 2 of the paper. Section 4 concludes the paper.

2 Preliminaries

There are many ways in which a random variable X can be said to be smaller than another random variable Y. In the usual stochastic ordering case, a random variable X with survival function $\bar{F} = 1 - F$ is stochastically smaller than a random variable Y with survival function $\bar{G} = 1 - G$, denoted by $X \leq_{st} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all x. For more details on various kinds of stochastic orders, one may refer to [7].

For a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ with the joint distribution function F and univariate marginal distribution functions $F_1, \ldots, F_n$, if there exists some $C : [0,1]^n \longrightarrow [0,1]$ such that, for all x_i , $i = 1, \ldots, n$,

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)),$$

then C is called as the copula of X. A real function ϕ is n-monotone on $(a,b) \subseteq \mathbb{R}$ if $(-1)^{n-2}\phi^{(n-2)}$ is decreasing and convex in (a,b) and $(-1)^k \phi^{(k)}(x) \geq 0$ for all $x \in (a,b), k = 0, 1, \ldots, n-2$, in which $\phi^{(i)}(.)$ is the ith derivative of $\phi(.)$. For a n-monotone $(n \geq 2)$ function $\phi : [0, +\infty) \longrightarrow [0,1]$ with $\phi(0) = 1$ and $\lim_{x \to +\infty} \phi(x) = 0$, let $\psi = \phi^{-1}$, be the right continuous inverse of ψ, then

$$C_\phi(u_1, \ldots, u_n) = \phi(\psi(u_1) + \ldots + \psi(u_n)), \quad \text{for all}\, u_i \in [0,1], i = 1, \ldots, n,$$

is called an Archimedean copula with generator ϕ. Archimedean copulas cover a wide range of dependence structures including the independence copula. For more detail on Archimedean copulas, readers may refer to [6].

Majorization orders are quite useful and powerful in establishing various inequalities. For preliminary notations and terminologies on majorization theory, see [5]. Let $\mathbf{x} = (x_1, \ldots, x_n)$

and $\mathbf{y} = (y_1, \ldots, y_n)$ be two real vectors and $x_{(1)} \leq \ldots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x}$.

Definition 1. The vector $\mathbf{x}$ is said to be

(i) weakly submajorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \preceq_{\mathrm{w}} \mathbf{y}$) if $\sum_{i=j}^{n} x_{(i)} \leq \sum_{i=j}^{n} y_{(i)}$ for all $j = 1, \ldots, n$,

(ii) weakly supermajorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \overset{\mathrm{w}}{\preceq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \ldots, n$.

Before proceeding to main results, let us present some lemmas to be utilized in the sequel. The first two lemmas concern majorization, Schur-convexity and Schur-concavity.

Lemma 2.1 ([5], Theorem 3.A.4)**.** Suppose $\mathbb{I} \subset \mathbb{R}$ is an open interval and $\Phi : \mathbb{I}^n \longrightarrow \mathbb{R}_+$ is continuously differentiable. Necessary and sufficient conditions for Φ to be Schur-convex (Schur-concave) on $\mathbb{I}^n$ are

(i) Φ is symmetric on $\mathbb{I}^n$,

(ii) for $i \neq j$ and all $z \in \mathbb{I}^n$,

$$(z_i - z_j)\left(\frac{\partial \Phi(z)}{\partial z_i} - \frac{\partial \Phi(z)}{\partial z_j}\right) \geq (\leq) 0,$$

where $\frac{\partial \Phi(z)}{\partial z_i}$ denotes the partial derivative of Φ with respect to its i-th argument.

Lemma 2.2 ([5], Theorem 3.A.8)**.** For a function l on $A \in \mathbb{R}^n$, $\mathbf{x} \preceq_{\mathrm{w}} (\overset{\mathrm{w}}{\preceq})\mathbf{y}$ implies $l(\mathbf{x}) \leq l(\mathbf{y})$ if and only if it is increasing (decreasing) and Schur-convex on A.

The following lower orthant order on Archimedean copulas will also be utilized in the sequel.

Lemma 2.3 ([4], Lemma A.1)**.** For two n-dimensional Archimedean copulas $C_{\phi_1}(\mathbf{u})$ and $C_{\phi_2}(\mathbf{u})$, if $\psi_2 \circ \phi_1$ is super-additive, then $C_{\phi_1}(\mathbf{u}) \leq C_{\phi_2}(\mathbf{u})$ for all $\mathbf{u} \in [0, 1]^n$.

3 Main result

[4] might be the first to investigate the ordering properties of order statistics from statistically dependent observations assembled with some kind of Archimedean copulas. In this section, we carry out stochastic comparisons between parallel systems consisting of interdependent heterogeneous GGD components assembled with some kind of Archimedean copula according to the usual stochastic order. Recall that a random variable X belongs to the Exponentiated Scale (ES) family of distributions if $X \sim H(x) = [G(\lambda x)]^{\alpha}$, where $\alpha, \lambda > 0$ and G is called the baseline distribution function which we assume that is absolutely continuous. In the sequel, we denote this family by ES(α, λ).

In the ES family, [2] obtained the following theorems for the comparison of parallel systems under the usual stochastic order.

Theorem 3.1. For $\boldsymbol{X} \sim \text{ES}(\boldsymbol{\alpha}, \lambda, \phi_1)$ and $\boldsymbol{X^*} \sim \text{ES}(\boldsymbol{\alpha^*}, \lambda, \phi_2)$,

(i) if ϕ_1 or ϕ_2 is log-convex, and $\psi_2 \circ \phi_1$ is super-additive, then $(\alpha_1, \ldots, \alpha_n) \succeq_{\text{w}} (\alpha_1^*, \ldots, \alpha_n^*)$ implies $X_{n:n} \geq_{\text{st}} X_{n:n}^*$;

(ii) if ϕ_1 or ϕ_2 is log-concave, and $\psi_1 \circ \phi_2$ is super-additive, then $(\alpha_1, \ldots, \alpha_n) \stackrel{\text{w}}{\succeq} (\alpha_1^*, \ldots, \alpha_n^*)$ implies $X_{n:n} \leq_{\text{st}} X_{n:n}^*$.

The following result follows immediately from Theorem 3.1.

Theorem 3.2. For $\boldsymbol{X} \sim \text{GGD}(\boldsymbol{\theta}, \alpha, \lambda, \phi_1)$ and $\boldsymbol{X^*} \sim \text{GGD}(\boldsymbol{\theta^*}, \alpha, \lambda, \phi_2)$,

(i) if ϕ_1 or ϕ_2 is log-convex, and $\psi_2 \circ \phi_1$ is super-additive, then $(\beta_1, \ldots, \beta_n) \succeq_{\text{w}} (\beta_1^*, \ldots, \beta_n^*)$ implies $X_{n:n} \geq_{\text{st}} X_{n:n}^*$;

(ii) if ϕ_1 or ϕ_2 is log-concave, and $\psi_1 \circ \phi_2$ is super-additive, then $(\beta_1, \ldots, \beta_n) \stackrel{\text{w}}{\succeq} (\beta_1^*, \ldots, \beta_n^*)$ implies $X_{n:n} \leq_{\text{st}} X_{n:n}^*$.

For the Archimedean survival copula, log-convexity of the generator leads to the RTIS (right tail increasing in sequence) property. Also, for many sub-families of Archimedean copulas, the superadditivity of $\psi_2 \circ \phi_1$ can be roughly interpreted as follows: Kendall's τ of the copula with generator ϕ_2 is larger than that with generator ϕ_1 and hence is more positive dependent.

Example 3.3. Suppose that X and X^* have either of the following two dependence structures. (i) Gumbel copulas with respective generators

$$\phi_1(x) = e^{-x^{\frac{1}{\beta_1}}}, \quad \phi_2(x) = e^{-x^{\frac{1}{\beta_2}}}, \beta_2 \geq \beta_1 \geq 1;$$

(ii) Archimedean copulas with respective generators

$$\phi_1(x) = (x^{\frac{1}{\beta_1}})^{-1}, \quad \phi_1(x) = (x^{\frac{1}{\beta_1}})^{-1}, \beta_2 \geq \beta_1 \geq 1.$$

It is easy to see that ϕ_i is log-convex for $i = 1, 2$. In view of $\psi_2(\phi_1(0)) = 0$ and the convexity of $\psi_2(\phi_1(x)) = x^{\frac{\beta_2}{\beta_1}}$, we conclude that $\psi_2(\phi_1(x))$ is super-additive by Proposition 21.A.11 in [5].

Theorem 3.4. For $\boldsymbol{X} \sim \text{GGD}(\theta, \alpha, \boldsymbol{\lambda}, \phi_1)$ and $\boldsymbol{X^*} \sim \text{GGD}(\theta, \alpha, \boldsymbol{\lambda^*}, \phi_2)$, ϕ_1 or ϕ_2 is log-convex, and $\psi_2 \circ \phi_1$ is super-additive, then $(\lambda_1, \ldots, \lambda_n) \stackrel{\text{w}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*)$ implies $X_{n:n} \geq_{\text{st}} X_{n:n}^*$.

Proof. $X_{n:n}$ and $X_{n:n}^*$ have their respective distribution functions, for $x \geq 0$,

$$F_{X_{n:n}}(x) = \phi_1\big(\sum_{i=1}^{n} \psi_1([1 - \exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x} - 1)\}]^{\theta})\big) = J(\boldsymbol{\lambda}, \alpha, \theta, x, \phi_1), \tag{3.1}$$

$$F_{X_{n:n}^*}(x) = \phi_2\big(\sum_{i=1}^{n} \psi_2([1 - \exp\{-\frac{\lambda_i^*}{\alpha}(e^{\alpha x} - 1)\}]^{\theta})\big) = J(\boldsymbol{\lambda^*}, \alpha, \theta, x, \phi_2). \tag{3.2}$$

We only prove the case that ϕ_1 is log-convex, and the other case can be finished similarly. Since ϕ_1 is decreasing, we have

$$\frac{\partial J(\boldsymbol{\lambda},\alpha,\theta,x,\phi_1)}{\partial\lambda_i}=\frac{(e^{\alpha x}-1)}{\alpha}\theta\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}[1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}]^{\theta-1}$$
$$\times\frac{\phi_1'\sum_{i=1}^n\psi_1([1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}]^{\theta}))}{\phi_1'\big(\psi_1([1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}]^{\theta})\big)}\geq 0,$$
$$\text{for all}\quad x>0,$$

That is, $J(\boldsymbol{\lambda},\alpha,\theta,x,\phi_1)$ is increasing in λ_i for $i=1,\dots,n$. Furthermore, for $i\neq j$,

$$\frac{\partial J(\boldsymbol{\lambda},\alpha,\theta,x,\phi_1)}{\partial\lambda_i}-\frac{\partial J(\boldsymbol{\lambda},\alpha,\theta,x,\phi_1)}{\partial\lambda_i}=$$
$$\frac{(e^{\alpha x}-1)}{\alpha}\theta\phi_1'\sum_{i=1}^n\psi_1([1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}]^{\theta}))$$
$$\Bigg(\frac{\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}}{1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}}\frac{\phi_1(\psi_1([1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}]^{\theta}))}{\phi_1'\big(\psi_1([1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}]^{\theta})\big)}-$$
$$\frac{\exp\{-\frac{\lambda_j}{\alpha}(e^{\alpha x}-1)\}}{1-\exp\{-\frac{\lambda_j}{\alpha}(e^{\alpha x}-1)\}}\frac{\phi_1(\psi_1([1-\exp\{-\frac{\lambda_j}{\alpha}(e^{\alpha x}-1)\}]^{\theta}))}{\phi_1'\big(\psi_1([1-\exp\{-\frac{\lambda_j}{\alpha}(e^{\alpha x}-1)\}]^{\theta})\big)}\Bigg).$$

Note that the log-convexity of ϕ_1 implies the decreasing property of $\frac{\phi_1}{\phi_1'}$. Since $\psi_1([1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}]^{\theta})$ is decreasing in $\lambda_i>0$, then
$\dfrac{\phi_1(\psi_1([1-\exp\{-\frac{\lambda_j}{\alpha}(e^{\alpha x}-1)\}]^{\theta}))}{\phi_1'\big(\psi_1([1-\exp\{-\frac{\lambda_j}{\alpha}(e^{\alpha x}-1)\}]^{\theta})\big)}$ is increasing in $\lambda_i>0$. Also the decreasing property of $\frac{\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}}{1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}}$, and thus
$\frac{\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}}{1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}}\dfrac{\phi_1(\psi_1([1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}]^{\theta}))}{\phi_1'\big(\psi_1([1-\exp\{-\frac{\lambda_i}{\alpha}(e^{\alpha x}-1)\}]^{\theta})\big)}$ is increasing in $\lambda_i>0$. So, for $i\neq j$,

$$(\lambda_i-\lambda_j)\Bigg(\frac{\partial J(\boldsymbol{\lambda},\alpha,\theta,x,\phi_1)}{\partial\lambda_i}-\frac{\partial J(\boldsymbol{\lambda},\alpha,\theta,x,\phi_1)}{\partial\lambda_j}\Bigg)\leq 0.$$

Then Schur-concavity of $J(\boldsymbol{\lambda},\alpha,\theta,x,\phi_1)$ follows from Lemma 2.1. According to Lemma 2.2 $(\lambda_1,\dots,\lambda_n)\overset{\text{w}}{\succeq}(\lambda_1^*,\dots,\lambda_n^*)$ implies $J(\boldsymbol{\lambda},\alpha,\theta,x,\phi_1)\leq J(\boldsymbol{\lambda^*},\alpha,\theta,x,\phi_1)$. On the other hand, since $\psi_2\circ\phi_1$ is super-additive by Lemma 2.3,
we have $J(\boldsymbol{\lambda^*},\alpha,\theta,x,\phi_1)\leq J(\boldsymbol{\lambda^*},\alpha,\theta,x,\phi_2)$. So, it holds that

$$J(\boldsymbol{\lambda},\alpha,\theta,x,\phi_1)\leq J(\boldsymbol{\lambda^*},\alpha,\theta,x,\phi_1)\leq J(\boldsymbol{\lambda^*},\alpha,\theta,x,\phi_2).$$

That is, $X_{n:n}\geq_{\text{st}}X^*_{n:n}$. $\square$

[1] proved the following general result.

Theorem 3.5. For $\boldsymbol{X} \sim \mathrm{ES}(\boldsymbol{\alpha}, \lambda, \phi_1)$ and $\boldsymbol{X}^* \sim \mathrm{ES}(\boldsymbol{\alpha}^*, \lambda, \phi_2)$, if $\psi_2 \circ \phi_1$ is super-additive, then $\boldsymbol{\alpha} \overset{\mathrm{w}}{\succeq} \boldsymbol{\alpha}^*$ implies $X_{1:n} \leq_{\mathrm{st}} X^*_{1:n}$.

The following corollary immediately follows from the above theorem.

Corollary 3.6. Suppose $\boldsymbol{X} \sim GGD(\alpha, \lambda, \boldsymbol{\theta}, \phi_1)$ and $\boldsymbol{X}^* \sim GGD(\alpha, \lambda, \boldsymbol{\theta}^*, \phi_2)$ and $\phi_2 \circ \psi_1$ is super-additive. Then $\boldsymbol{\beta} \overset{\mathrm{w}}{\succeq} \boldsymbol{\beta}^*$ implies $X_{1:n} \leq_{\mathrm{st}} X^*_{1:n}$.

4 Conclusions

In this paper, in the presence of the Archimedean copula or survival copula for the random variables of samples having generalized Gompertz distribution, we obtain the usual stochastic order of the sample extremes. In addition, some examples illustrating the main results are presented as well.

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Application of D-Vine Regression Copula in Covid-19 Data

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Abstract

The rapid spread of Covid-19 since January 2020 has dramatically affected financial markets and economies all over the world, especially in United States. This paper aims at utilizing the regression model of D-Vine Copula to investigate about the effects of each input variables related to coronavirus news on our three response variables which are three famous indices in U.S. Findings demonstrate that the fitted quantile curves of all input variables suggest that the news variables have the most negative effect on all mentioned indices.
Keywords: Pandemic, Covid-19, indices, D-Vine copula, Kendall's tau

1 Introduction

Everything started in December 2019. The coronavirus disease also known as COVID-19 was first reported from Wuhan, China. Not only this virus with mortality rate of 3% has affected the families adversely in over 120 countries, but also it has slowed the monetary and financial markets in all over the world. On May 11, president trump announces the suspension of all travel from *Europe*

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excluding United Kingdom to the United States for the next 30 days. In just one day, three of top's Wall Street indices, S&P 500, NASDAQ 100 and Dow jones fell more than 9 percent. Black Thursday, March 12 was marked as the worst day of stock exchange market in 21th century. These indices have lost their worth nearly 20 percent in the last two weeks. US stocks have lost $11.5 trillion in one month since February 19th.

Vine Copula model and its classes has a variety of functions and long has been used in determining the independent data for multivariate data structures. This class of flexible copula models has become very well known in the recent years for many applications in various fields such as finance and engineering. The popularity of vines copulas is due to the fact that it allows in addition to the separation of margins and dependence by the copula approach, tail asymmetries, and separate multivariate component modeling. In 2010, Joe et all, described tail dependence and conditional tail dependence and also its probabilities[6]. After that, Nikoloulopoulos and et al in 2012, published a paper about Vine Copulas with symmetric tail dependence and its applications to financial return data[9]. Two years later in 2014, So and Yeung, wrote a paper "Vine-Copula GARCH model with dynamic conditional dependence". They developed a generic approach to specifying dynamic conditional dependence using any dependence measures [12]. Finally, Kraus and Czado in 2016, Conducted a research about D-Vine copula based quantile regression [7]. On the other hand, some studies have been done regarding the pandemic's effect on financial markets. In 2020, Corbet et al, indicated that the volatility relationship between the main Chinese stock markets and Bitcoin evolved significantly during the period of epidemic [4]. Later in the same year, Zhang et al. [13]. Then again in 2020, Ali et al. [2] have done some research in this area. This paper is organized as follows. In section 2 we explain that how and where we achieved the data and using them in analyzing the situation of the 3 popular indices. Section 3 is dedicated to the explanation of Vine Copula models which we use for our analysis in the next section. Finally, the last section, section 5 has to do with the discussion and conclusion of the paper.

2 Data

We would like to investigate the effects of coronavirus regarding the number of cases and related financial news on three indices since the beginning of year 2020 until the end of May. We have two sets of data. Financial news which are released in specific time in the most popular websites and also coronavirus cases in United States from the moment that the first case was reported. Both of the data's date are from January 1$^{\text{st}}$ to the end of May (we collected data for 5 months). We acquired data of cases from the website of Worldometers and financial news from finance-related websites such as: Yahoo Finance, Bloomberg, Harvard Business Review, Department of treasury and Bureau of Economic Analysis using text mining. We searched those keywords and collected the first 1000 results for each keyword from January 1$^{\text{st}}$ to May 31$^{\text{st}}$. In order to obtain a more robust dataset, we corrected the results by filtering only those results that had reported a publication date between that period and then removed duplicate results. It gave us a total number of 397 web pages from those websites.

3 Vine Copula Models

The n-dimensional vine copulas are built via mixing from $n(n-1)/2$ bivariate linking copulas on trees and their copula density functions. Since the densities of multivariate vine copulas can be factorized in terms of bivariate copulas and lower-dimensional margins. Depending on the type of trees, various vine copulas can be constructed. Two boundary cases are D-vines and C-vines. (see Bedford and Cooke (2002, 2001)[2] [3], Kurowicka and Cooke (2006) [8] and Section 4.5 of Joe (1997)[5]). For D-vines the density is given as equations 3.1 (Aas et al. (2009)[1]),

$$f(x_1,\dots,x_n)=\prod_{k=1}^{n} f_k(x_k)\prod_{j=1}^{n-1}\prod_{i=1}^{n-j} c_{i,i+j|i+1,\dots,i+j-1}\left(F_{i|i+1,\dots,i+j-1},F_{i+j|i+1,\dots,i+j-1}\right) \tag{3.1}$$

Index j denotes the tree/level, while i runs over the edges in each tree.

3.1 D-vine based quantile regression model

The main purpose of D-vine copula based quantile regression is to predict the quantile of a response variable Y given the outcome of some predictor variables $X_1,\dots,X_n$, where $Y\sim F_y$ and $X_j\sim F_j; j=1,\dots,n$. Hence, the focus of interest lies on the joint modeling of Y and X and in particular on the conditional quantile function for $\alpha\epsilon(0,1)$:

$$q_\alpha(X_1,\dots,X_n)=F^{-1}_{Y|X_1,\dots,X_n}(\alpha|x_1,\dots,x_n) \tag{3.2}$$

Using the probability integral transforms

$$F^{-1}_{Y|X_1,\dots,X_n}(\alpha\,|x_1,\dots,x_n)=F_Y^{-1}(C^{-1}_{V|U_1,\dots,U_n}(\alpha\,|u_1,\dots,u_n)) \tag{3.3}$$

Now, we can obtain an estimate of the conditional quantile function by estimating the marginal F_y and $F_j; j=1,\dots,n.$ as well as the copula $C_{V|U_1,\dots,U_n}$ and plugging them into bellow Equation

$$\widehat{q_\alpha}(X_1,\dots,X_n)=\widehat{F^{-1}_{Y|X_1,\dots,X_n}}\left(\widehat{C^{-1}_{V|U_1,\dots,U_n}}(\alpha\,|\widehat{u_1},\dots,\widehat{u_n})\right). \tag{3.4}$$

For more details see [7].

4 Analysis

For investigating the effect of three input data on three well known indices, the regression model of D-Vine Copula is implemented. First, for time series data, we need to remove the serial dependence which is present in each component. This will be accomplished by using standard univariate financial time series models such as the class of GARCH models. For each of variables, we fit a GARCH(1,1) model with standardized student's t-distribution innovation. Therefore, we use the cumulative distribution function of the standardized Student's t-distribution to define the copula data as probability integral transform. First, before fitting D-vine copula, we investigate the pairwise dependencies among the representatives of each variables. From the contour shapes, we observe the evidence of negative correlation between news and three indices of U.S exchange markets and a high positive correlation between three indices of U.S exchange markets and also very low correlation exists between financial news and number of deaths. We investigated three D-vine regression specifications. In each model, response variable is one of three indices of U.S exchange markets (S&P 500, NASDAQ 100, Dow Jones) and predictive variables are the number of cases, deaths and financial news. Parameters are estimated using the sequential estimation method. Table 1 demonstrates the selected covariates and their ranking order in the D-vine, the copula families associated to each trees, sequential parameters estimates, as well as implied Kendall's tau (allowing for all implemented pair copula families), and log –likelihood values for each models.

Table 1

D-vine order: S&P500| deaths, cases, news

$1 --> S.P$, $2 --> case$, $3 --> dead$, $4 --> news$

tree	*edge*	*copula*	*par*	*par2*	*tau*	*log − likelihood*
1	1,3	t	0.026	3.077	0.017	1.57
1	3,2	bb6	1.5	1.1	0.278	14.18
1	2,4	bb8	1.12	0.99	-0.058	0.78
2	1,2:3	bb8	7.22	0.16	-0.122	2.01
2	3,4;2	bb8	1.1	1.0	-0.046	0.88
3	1,4;2,3	bb8	1.11	0.99	-0.053	0.73

Log-lik= -2.49, **Aic** = 24.21,**Bic** = 49.37

D-vine order: NASDAQ100| news, deaths, cases

$1 --> NASDAQ$, $2 --> case$, $3 --> dead$, $4 --> news$

selected covariates, implied Kendall's tau and log–likelihood values for each model As you can see, all the predictive variables are included in D-Vine regression model meaning that all three input variables are effective on response variables (three indices).

5 Discussion

Ultimately, it can be inferred that all the predictive variables are included in D-Vine regression model which implies that all three input variables are effective on response variables (three indices). the findings indicate that the fitted quantile curves of all input variables suggest that number of death has the most negative effect on S&P500 and Dow Jones and the variable news has the most negative influence on NASDAQ100 and also intuitively it can be concluded that variable D (GDP news) and B (recession) then F (pay check) have the most effect on all mentioned indices respectively.

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A New Family of Copulas Based on Distortion Function and its Properties

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Abstract

One of the basic methods of constructing the copula function is to use the bivariate survival function. In this article, using the copula function and taking into properties of distortion functions, a family of distribution functions is introduced and its characteristics are investigated. In the following, using the introduced distribution function, a bivariate survival function is presented and based on it, a family of copula functions is introduced.
Keywords: Copula Function , Distortion Function, Dependence Structure, Dependence Measures.

1 Introduction

One of the most important tools for linking two random variables is the copula functions. These functions that used to link the univariate marginal distribution functions and their corresponding common distribution function, first introduced in [9]. Sklar's theorem proves the existence of a unique copula that captures the dependence structures among continuous random variables. These

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functions have many applications in the field of probability and statistics . The most important and complete references for studying the copulas, their features, and applications are [8] and [6].

Distortion functions aimed at converting standard distribution functions for the premium calculation was introduced by [11] and [10] are useful tools in generalizing of standard distribution functions. Distorted measures have been used in pricing of insurance contracts for a long time. [3] showed that distortion is a well known premium calculation principle for insurance contracts.

In the following, the definition of the copula and distortion functions and its basic features as well as some concepts required in the article are stated.

Definition 1. *A function $C : [0,1]^2 \to [0,1]$ is a copula function if for all $u, v \in [0,1]$, the following properties hold:*

i) $C(u,0) = C(0,v) = 0$.

ii) $C(u,1) = u, \quad C(1,v) = v$.

iii) For every $u_1, u_2, v_1, v_2 \in [0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$, we have

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u1,v1) \geq 0.$$

Definition 2. *A continuous increasing function $g : [0,1] \longrightarrow [0,1]$ such that $g(0) = 0$ and $g(1) = 1$ is called distortion function.*

The most famous measures of dependence are Kendall's τ, Spearman's ρ, Blomqvist's β, and Gini's γ, which are defined as follows according to copula function $C(u,v)$ (see [8]).

Definition 3. *Let $C(u,v)$ be a copula function. The measure of dependence for it are:*

$$\begin{aligned}\tau_C &= 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1 \\ &= 1 - 4\int_0^1\int_0^1 \frac{\partial C(u,v)}{\partial u}\frac{\partial C(u,v)}{\partial v}dudv, \end{aligned} \tag{1.1}$$

$$\rho_C = 12\int_0^1\int_0^1 C(u,v)dudv - 3, \tag{1.2}$$

$$\beta_C = 4C(\frac{1}{2},\frac{1}{2}) - 1, \tag{1.3}$$

$$\gamma_C = 4[\int_0^1 C(u,1-u)du - \int_0^1 [u - C(u,u)]du]. \tag{1.4}$$

In this paper, we present a new family copula based on the distortion distribution and examine its features. For this purpose, we first introduce a family of univariable distortion distributions and study its reliability features such as hazard rate function and aging intensity. Then, based on this family, we introduce a family of bivariate distributions and examine its characteristics. In the following, using the bivariate distribution family, we define a new copula function and state some dependence structure for it.

2 Main results

2.1 A family of bivariate distributions

In this section, we first introduce a family of distribution functions and then present a family of bivariate distributions based on it. For this aim, we know tha if $C(u, v)$ be a copula function, then for every $\xi \in (0, 1]$ the function $g_\xi(t) = \frac{C(\xi,t)}{\xi}$ for all $t \in [0, 1]$ is a distortion, because $g_\xi(0) = 0$ and $g_\xi(1) = 1$ and also

$$g_\xi^{'}(t) = \frac{\partial g_\xi(t)}{\partial t} = \frac{\partial C(\xi, t)}{\xi \partial t} = P[V \leq t|U = \xi] \geq 0,$$

therefore $g_\xi(t)$ is increasing. See [7]. In the following, we provide a new family of distribution.

[1] introduced a new practical way of generating comprehensive copula. One of the copula presented by him is,

$$C(u, v) = uv \exp\{\eta(1 - u)(1 - v)\}, \quad |\eta| \leq 1. \tag{2.1}$$

[2] and [4] have also presented interesting results by studying copula (2.1) and examining its properties.

It is simply clear that, considering the copula (2.1), the function

$$g_\xi(t) = t \exp[\eta(1 - \xi)(1 - t)], \tag{2.2}$$

is a distortion function. Now, using this distortion function, we introduce a family of univariate distributions.

Proposition 2.1. *Let in distortion function (2.2) for negative η, put $-\theta = \eta(1 - \xi)$. Then, we have*

$$g(t) = t \exp\{-\theta(1 - t)\}, \quad 0 \leq t \leq 1. \tag{2.3}$$

It is explicit (2.3) is a increasing and convex function. Also (2.3) is a distortion function with $g(0) = 0$ and $g(1) = 1$. Now if $F(.)$ be a distribution function of a random variable then for $t = F(x)$,

$$F_g(x) = g(F(x)) = F(x) \exp(-\theta \bar{F}(x)), \tag{2.4}$$

is named the distorted distribution function of distribution function $F(x)$ and also $Fg(x)$ is the lifetime of the parallel system introduced above.

It is clear that by changing distribution $F(.)$, a new distorted distribution is produced. Therefore, (2.4) is a general form of a family distorted distributions and, it can be concluded that $S(t) = 1 - F_g(t)$ is a survival function. Note that $S(t)$ is the survival function of the univariate distortied distribution.

2.1.1 Reliability index

Some reliability indicators such as hazard function, ageing intensity for (2.4) are described below.

Remark 2.2. Let X be a random variable with distribution function (2.4). Then the hazard rate function of X consists of

$$h_g(t) = \frac{f_g(t)}{\bar{F}_g(t)} = \frac{f(t)\exp(-\theta\bar{F}(t))(1+\theta F(t))}{1-F_g(t)}.$$

Jiang et al. (2003) showed that the representation of aging of a system by failure rate is qualitative, and accordingly, they introduced a new notion called aging intensity (AI). Let X be a absolutely continuous random variable with the distribution function F. The AI for X at time t, denoted by $L_X(t)$, is defined by $L_X(t) = \frac{h_X(t)}{H_X(t)}$, where $H_X(t) = \frac{1}{t}\int_0^t h_X(x)dx$ is the hazard rate average.

Remark 2.3. Let X be a random variable with distribution function (2.4). Then the AI of X consists of

$$L_g(t) = \frac{\frac{f(t)\exp(-\theta\bar{F}(x))(1+\theta F(t))}{1-F_g(t)}}{\int_0^t \frac{f(x)\exp(-\theta\bar{F}(x))(1+\theta F(x))}{1-F_g(x)}dx}.$$

Due to the complex shape of the presented distortion distribution, it is not possible to analyze the behavior of the hazard rate function and the AI of this distribution analytically, and for this, it is necessary to use its graph and application software.

2.2 A family of copula functions

One of the methods of constructing bivariate distributions is to use the univariate survival function. For this purpose, using model (2.4), we introduce a bivariate survival model and study its characteristics.

Proposition 2.4. *Let $S(t)$ be continuous survival function of model (2.4). In this case, for $t = x + y + \alpha xy, \alpha \in [0,1], \theta \geq 0$ the function $R : [0,\infty]^2 \longrightarrow [0,1]$, defined by*

$$R(x,y) = S(x+y+\alpha xy)$$

$$= 1 - F(x+y+\alpha xy)\exp(-\theta\bar{F}(x+y+\alpha xy)), \tag{2.5}$$

is a family of bivariate survival function.

Proof. Considering the conditions of bivariate survival functions, stated in Joe (2014), $R(0,0) = 1$, $R(x,\infty) = R(\infty,y) = R(\infty,\infty) = 0$, and $R(x,y)$ applies to the rectangle inequality if $\frac{\partial^2 R(x,y)}{\partial x \partial y} \geq 0$. It can be easily written that $R(x,0) = P(X > x, Y > 0) = S(x)$ and $R(0,y) = P(X > 0, Y > y) = S(y)$. $\square$

Now, using the Sklar's theorem, we obtain the family of copula function based on proposed bivariable distribution and calculate the dependence coefficients for it. For this purpose, we have

$$R(x, y) = \hat{C}(S(x), S(y)), \tag{2.6}$$

where $\hat{C}(\cdot, \cdot)$ is the survival copula. By using the transformations $u = S(x)$ and $v = S(y)$, in view of Sklar's theorem, we have

$$\hat{C}(u, v) = R(S^{-1}(u), S^{-1}(v)), \qquad u, v \in (0, 1).$$

Let $\psi(t; \theta) = \ln S(t) = \ln[1 - F(t) \exp(-\theta \bar{F}(t))]$, which is a decreasing function with $\psi(0) = 0$ and its derivative $\psi'(t) = \frac{-(1+\theta F(t)) f(t) \exp(-\theta \bar{F}(t))}{1-F(t) \exp(-\theta \bar{F}(t))}$ is a monotone function. Thus it can be written as $u = S(t) = \exp(\psi(t))$ and

$$S^{-1}(u) = \psi^{-1}(\ln(u)).$$

As a result

$$\hat{C}(u, v) = R(\psi^{-1}(\ln(u)), \psi^{-1}(\ln(v)), \qquad u, v \in (0, 1), \tag{2.7}$$

and thus

$$\hat{C}(u, v) = 1 - F(\psi^{-1}(\ln(u)) + \psi^{-1}(\ln(v)) + \alpha\psi^{-1}(\ln(u))\psi^{-1}(\ln(v)))$$

$$\times \exp\{-\theta \bar{F}[\psi^{-1}(\ln(u)) + \psi^{-1}(\ln(v)) + \alpha\psi^{-1}(\ln(u))\psi^{-1}(\ln(v))]\}. \tag{2.8}$$

According to the relationship between copula and survival copula functions $\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$ presented in [8], the copula corresponding to (2.8) is given by

$$\begin{aligned} C(u, v) = u + v - F(\psi^{-1}(\ln(1 - u)) + \psi^{-1}(\ln(1 - v)) \\ +\alpha\psi^{-1}(\ln(1 - u))\psi^{-1}(\ln(1 - v))) \\ \times \exp\{-\theta \bar{F}[\psi^{-1}(\ln(1 - u)) + \psi^{-1}(\ln(1 - v)) \\ +\alpha\psi^{-1}(\ln(1 - u))\psi^{-1}(\ln(1 - v))]\}. \end{aligned} \tag{2.9}$$

We call the copula function (2.9) as the distortion family of copula and denoted by DFC.

2.3 Some measures of dependence

Now, to investigate the behavior of DFC, we derive some nonparametric measure of dependence for this model using the definition 3 in Section 1.

Proposition 2.5. *If (X, Y) is a random vector with corresponding DFC, then the measures of dependence for (X, Y) with DFC are given below:*

$$\begin{aligned}\tau_{DFC} =& 1-4\int_0^1\int_0^1 \{1-\frac{\partial K(u,v)}{\partial u} f(K(u,v))\exp[-\theta \bar{F}(K(u,v))](1+\theta F(K(u,v)))\} \\ & \times\{1-\frac{\partial K(u,v)}{\partial u} f(K(u,v))\exp[-\theta \bar{F}(K(u,v))](1+\theta F(K(u,v)))\} dudv,\end{aligned}$$

where $K(u,v)=\psi^{-1}(\ln(1-u))+\psi^{-1}(\ln(1-v))+\alpha\psi^{-1}(\ln(1-u))\psi^{-1}(\ln(1-v))),$

$$\begin{aligned}\rho_{DFC} =& 12\int_0^1\int_0^1 \{u+v-F(\psi^{-1}(\ln(1-u))+\psi^{-1}(\ln(1-v)) \\ & +\alpha\psi^{-1}(\ln(1-u))\psi^{-1}(\ln(1-v))) \\ & \times\exp\{-\theta\bar{F}[\psi^{-1}(\ln(1-u))+\psi^{-1}(\ln(1-v)) \\ & +\alpha\psi^{-1}(\ln(1-u))\psi^{-1}(\ln(1-v))]\} dudv-3,\end{aligned}$$

$$\begin{aligned}\gamma_{DFC} =& 4\int_0^1 \{1-F(\psi^{-1}(\ln(1-u))+\psi^{-1}(\ln(u)) \\ & +\alpha\psi^{-1}(\ln(1-u))\psi^{-1}(\ln(u))) \\ & \times\exp\{-\theta\bar{F}[\psi^{-1}(\ln(1-u))+\psi^{-1}(\ln(u)) \\ & +\alpha\psi^{-1}(\ln(1-u))\psi^{-1}(\ln(u))]\} du, \\ & +4\int_0^1 \{u-F[2\psi^{-1}(\ln(1-u))+\alpha(\psi^{-1}(\ln(1-u)))^2] \\ & \times\exp\{-\theta\bar{F}[2\psi^{-1}(\ln(1-u))+\alpha(\psi^{-1}(\ln(1-u)))^2]\} du\end{aligned}$$

$$\begin{aligned}\beta_{DFC} =& 3-4F(\psi^{-1}(\ln(\frac{1}{2}))+\psi^{-1}(\ln(\frac{1}{2}))+\alpha\psi^{-1}(\ln(\frac{1}{2}))\psi^{-1}(\ln(\frac{1}{2}))) \\ & \times\exp\{-\theta\bar{F}[\psi^{-1}(\ln(\frac{1}{2}))+\psi^{-1}(\ln(\frac{1}{2}))+\alpha\psi^{-1}(\ln(\frac{1}{2}))\psi^{-1}(\ln(\frac{1}{2}))]\}.\end{aligned}$$

In the following examples, considering some special distributions, we introduce some new copula functions.

Example 2.6. Let X be a exponential random variable with parameter λ. Using relations (2.4) and (2.9), a new copula can be written as follows, and we call it distortion exponential copula.

$$\begin{aligned}C_\lambda(u,v) =& u+v-[1-\exp\{-\lambda(\psi^{-1}(\ln(1-u))+\psi^{-1}(\ln(1-v)) \\ & +\alpha\psi^{-1}(\ln(1-u))\psi^{-1}(\ln(1-v)))\}]\exp\{-\theta\exp[-\lambda(\psi^{-1}(\ln(1-u)) \\ & +\psi^{-1}(\ln(1-v))+\alpha\psi^{-1}(\ln(1-u))\psi^{-1}(\ln(1-v)))]\}.\end{aligned}$$

Example 2.7. Let X be a weibull random variable with shape parameter η and scale parameter β. Using relations (2.4) and (2.9), a new copula can be written as follows, and we call it distortion exponential copula.

$$C_{\eta,\beta}(u,v) = u + v - [1 - \exp\{-(\frac{K(u,v)}{\beta})^{\eta}\}] \exp\{-\theta \exp(-(\frac{K(u,v)}{\beta})^{\eta})\},$$

where in

$$K(u,v) = \psi^{-1}(\ln(1-u)) + \psi^{-1}(\ln(1-v)) + \alpha\psi^{-1}(\ln(1-u))\psi^{-1}(\ln(1-v))).$$

3 Conclusions

In this article, we introduced the new family of distortion distribution and studied its reliability properties. Then we presented a new class of bivariate distributions and so on offered new family of copula functions and its properties based on bivariate distribution. The dependence structure of this family of copulas, as well as the tail dependence for it, has been studied. Some measures of dependency were also obtained for the introduced copula.

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Semiparametric estimation of mutual information for elliptical copulas

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Abstract

Mutual information can be rewritten based on the copula density and considered as a dependency measure. In this paper, a semiparametric estimation of this measure based on the probit transformation method is presented. A simulation study is performed to measure the accuracy of the estimators on elliptical copulas. The simulation results show that the suggested method has better performance than beta kernel and Bernstein methods.
Keywords: Mutual information, Copula density, Probit transformation.

1 Introduction

Understanding and modelling dependence in multivariate relationships has a pivotal role in scientific investigations. Pearson's coefficient measures linear correlation under the assumption of normality of the marginal distributions. These conditions are not always provided and Spearman's rho and Kendall's tau coefficients are used instead.

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On the other hand, divergence measures are also used to measure the dependence between variables. Relative entropy, also known as Kullback–Leibler divergence, is derived from information theory and was originally introduced as a measure of the deviation of two probability distributions. Mutual information (MI) is a special case of relative entropy. Its original definition is based on the Kullback–Leibler (KL) divergence between the joint density of the random vector and the product of marginal densities.

Copulas provide a useful way to model different types of dependence structures explicitly. Instead of having one correlation number that encapsulates everything known about the dependence between two variables, copulas capture information on the level of dependence. Student-T (T) copula is a useful distribution in the elliptical copula class. Demarta and McNeil [3] showed that T copula is generally superior to the Gaussian copula in the context of modelling multivariate financial return data, because this copula able to capture the tail dependence among extreme values.

Mutual information can be written as a function of the copula density and thus does not depend on its marginal distributions. Ma and Sun [5] introduced the concept of copula entropy by combining MI and the copula density. They demonstrated that the MI is equal to the negative of copula entropy. This measure was considered as a measure of multivariate association by Blumentritt and Schmid [1]. They provided the MI based on the Gaussian and T copulas. Recently, Mohammadi et al. [6] used copula based Jeffrey and Hellinger divergences as dependence measures. They show that the copula based Hellinger distance performs better than Kullback-Libeler divergence for small sample size or weak dependence.

Statistical estimation of mutual information has been considered by various authors. This paper deals with the semiparametric estimation of the MI based on copula density. Therefore, no marginal densities have to be estimated. Estimation is performed by kernel-based methods. In this paper, we focus on the local likelihood probit-transformation ($\mathcal{LLPT}$) method, which is the most recent method for copula density estimation. This method is used by Mohammadi et al. [7] for testing bivariate independence based on alpha-divergence. We compare the bias and root mean squared error (RMSE) of the corresponding estimators of MI using Monte Carlo simulation.

This paper provides a general framework for estimating the copula-based mutual information for T copula. These method is based on improved probit transformation method for copula density estimation. The rest of the paper is arranged as follows. In Section 2, the preliminaries for copula function copula density estimation using local likelihood probit transformation method are described. A semiparametric estimation of copula based mutual information for Gaussian and T copulas is provided in Section 3. In Section 4, a simulation study is performed to measure the accuracy of the suggested estimators.

2 Copula function

Some definitions related to a copula functions will be briefly reviewed based on Nelsen [9]. Let (X, Y) be a continuous random variable with joint cumulative distribution function (CDF) F, then copula C corresponding to F defined as:

$$F(x, y) = C(F_X(x), F_Y(y)), \qquad (x, y) \in R^2, \tag{2.1}$$

where F_X and F_Y are the marginal CDFs of X and Y, respectively. A bivariate copula function C is the CDF of random vector (U, V), defined on the unit square $[0,1]^2$, with uniform marginal distributions as $U = F_X(X)$ and $V = F_Y(Y)$.

The authors shall write $C(u, v; \theta)$ for a family of copulas indexed by the parameter θ. If $C(u, v; \theta)$ is an absolutely continuous copula distribution on $[0,1]^2$, then its density function is $c(u, v; \theta) = \frac{\partial^2 C(u,v;\theta)}{\partial u \partial v}$. As a result, the relationship between the copula density function (c) and the joint density function $f_{X,Y}(\cdot,\cdot)$ of random vector (X, Y) according to equation (2.1) can be represented as

$$f_{X,Y}(x, y) = c(F_X(x), F_Y(y); \theta) f_X(x) f_Y(y), \qquad (x, y) \in R^2, \tag{2.2}$$

where $f_X(\cdot)$ and $f_Y(\cdot)$ are the marginal density functions of X and Y, respectively.

2.1 Copula density estimation

The estimation of the copula density is needed to estimate the copula based mutual information. A specific class of nonparametric copula density estimators is kernel estimators. Charpentier et al. [2] presented different approaches to non-parametric estimation of the copula density such as mirror-reflection method, beta kernel, and transformation technique. In this paper, local likelihood probit-transformation ($\mathcal{LLPT}$) method is used to estimate the copula density suggested by Geenens et al. [4]. This method yields very good and easy to implement estimators and fixing boundary issues. The simple idea in $\mathcal{LLPT}$ method is to transform the domain of data to R^2. Then, standard kernel techniques can be used to estimate the density in R^2. Therefore a back-transformation yields an estimate of the copula density.

Let $(U_i, V_i)_{i=1,\ldots,n}$ are independent and identically distributed observations from the bivariate copula C. Then $(S_i, T_i) = (\Phi^{-1}(U_i), \Phi^{-1}(V_i))$ is a random vector with Gaussian margins and copula C. Thus, according to (2.2), an estimation of the copula density function can be given by

$$\hat{c}_n^{(\mathcal{PT})}(u, v) = \frac{\hat{f}_n(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}, \qquad (u, v) \in (0,1)^2. \tag{2.3}$$

In $\mathcal{LLPT}$ method $\hat{f}_n$ in (2.3) estmatted using locally fited a polynomial to the log-density of the transformed sample. Recently, Nagler [8] with a comprehensive simulation study has shown that $\mathcal{LLPT}$ method for copula density estimation yields very good. When the underlying density is on $[0,1]^2$, the performance of the kernel estimator depends on the choice of the kernel function and the bandwidth (smoothing parameter). For bandwidth choice, a practical approach is to consider the minimization of the AMISE on the level of the transformed data; see [4].

Gaussian (Normal) and Student-T (T) copulas belong to the Elliptical copula class. These copulas are very useful in financial data analysis.

2.2 Gaussian copula

The bivariate Gaussian copula can be constructed using a bivariate standard normal distribution and applying the of Sklar's theorem. The bivariate Gaussian copula with parameter ρ defined as:

$$\begin{aligned} C_G(u,v;\rho) &= \Phi_2(\Phi^{-1}(u),\Phi^{-1}(v);\rho) \\ &= \int_0^u \int_0^v c_G(s,t;\rho)dsdt,, \quad (u,v)\in[0,1]^2, \quad \rho\in[-1,1], \end{aligned}$$

where Φ_2 is the bivariate normal distribution function with zero mean vector, unit variances, and correlation ρ and Φ^{-1} denotes the quantile function of the univariate standard normal distribution. Moreover, c_G is the corresponding Gaussian copula density that using (2.2) can be expressed as

$$\begin{aligned} c_G(u,v;\rho) =& \frac{1}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))\sqrt{1-\rho^2}} \\ & \times \exp\Big\{\frac{2\rho\Phi^{-1}(u)\Phi^{-1}(v)-\rho^2(\Phi^{-1}(u)^2+\Phi^{-1}(v)^2)}{2(1-\rho^2)}\Big\}, \end{aligned}$$

where $(u,v)\in[0,1]^2$, $\rho\in[-1,1]$,, and ϕ denotes the density of univariate standard normal distribution.

2.3 Student's t copula

The random variables (X,Y) has bivariate standard Student's t distribution with correlation parameter ρ and $\nu>0$ degree of freedom (df) if its density is given by

$$f_t(x,y;\rho,\nu) = \frac{\Gamma(\frac{\nu+2}{\nu})(1-\rho^2)^{-1/2}}{\Gamma(\frac{\nu}{2})\nu\pi}\big(1+\frac{1}{\nu}\frac{x^2-2xy\rho+y^2}{1-\rho^2}\big)^{-\frac{\nu+2}{2}}. \tag{2.4}$$

The contour lines for a bivariate Gaussian as solid lines with $\rho=0.8$ and a bivariate standard Student's t distribution as dotted lines with $\rho=0.8,\nu=3$ (left panel) and $\rho=0.8,\nu=30$ (right panel) are given in Figure 1. We see that for small contour levels, the bivariate standard Student's t-density is larger than the bivariate standard Gaussian density, thus the bivariate standard Student's t distribution has heavier tails than the bivariate standard Gaussian distribution. As the degree of freedom increases, the contour lines of bivariate standard Gaussian and standard Student's distributions get closer together.

The bivariate Student's t (T) copula can be constructed using the bivariate standard Student's t distribution with ν degrees of freedom, correlation ρ as in equation (2.4) and is given as

$$\begin{aligned} C_T(u,v;\rho,\nu) &= T_{2,\nu}(T_\nu^{-1}(u),T_\nu^{-1}(v);\rho) \\ &= \int_0^u \int_0^v c_T(s,t;\rho,\nu)dsdt,, \quad (u,v)\in[0,1]^2, \quad \rho\in[-1,1],\nu>0. \end{aligned}$$

where $T_{2,\nu}$ is the bivariate Student's t distribution function with zero mean vector, unit variances, and correlation ρ and degrees of freedom ν. Moreover, T^{-1} denotes the quantile function of a

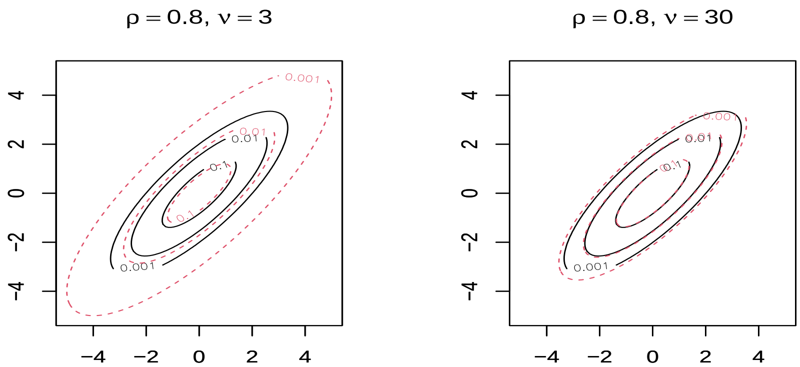

Figure 1: Bivariate contour lines for bivariate normal (solid lines) with $\rho = .8$ and bivariate standard Student's t (dotted lines) with $\rho = 0.8, \nu = 3$ (left panel) and $\rho = 0.8, \nu = 30$ (right panel)

standard Student's t distribution. We also give the expression of the bivariate T copula density function c_T as follows:

$$c_T(u, v; \rho, \nu) = \frac{f_t(T_\nu^{-1}(u), T_\nu^{-1}(v); \rho, \nu)}{t_\nu(T_\nu^{-1}(u))t_\nu(T_\nu^{-1}(v))}.$$

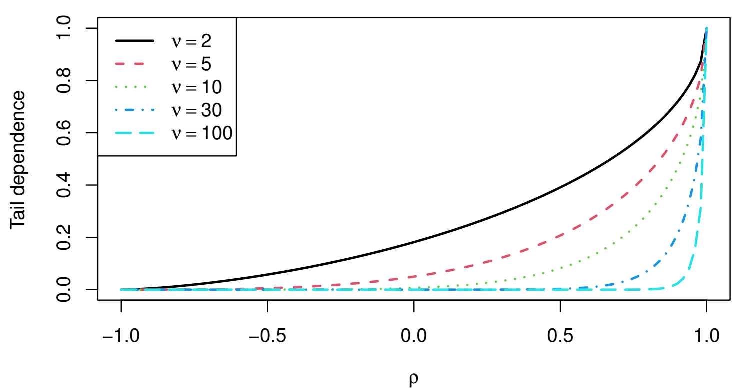

Figure 2: behavior of T copula with respect to parameters ρ and ν

The behavior of T copula with respect to parameters ρ and ν is presented in Figure 2. The lower and upper tail dependence for T copula with zero correlation parameter ($\rho = 0$) and two and five degrees of freedom ($\nu = 2, 5$) are equal to 0.182 and 0.05, respectively. In T copula with positive dependency and small degrees of freedom ($\nu < 10$) tail dependency occurs in both lower and upper tails and as the degree of freedom increases, dependency in the tail areas decreases.

3 Estimation of copula based mutual information

The idea of divergence measure has been widely employed in probability, statistics, information theory, and related fields. Kullback-Leibler (KL) divergence is a non symmetrical measure of the distinction between two probability density functions f_1 and f_2 defined as

$$KL(f_1 \parallel f_2) = \int_{-\infty}^{\infty} f_1(x) \log \frac{f_1(x)}{f_2(x)} dx, \qquad \alpha > 0, \alpha \neq 1.$$

This divergence is nonnegative, and $KL(f_1 \parallel f_2) = 0$ if and only if $f_1(x) = f_2(x)$.

The KL divergence between the joint density function and the product of marginal density functions is equivalent to mutual information (MI). The MI using equation (2.2) can be written as

$$\begin{aligned} MI(X,Y) &= KL(f \parallel f_X f_Y) \\ &= \int_{\mathbb{R}^2} f(x,y) \log\Big(\frac{f(x,y)}{f_X(x) f_Y(y)}\Big) dxdy \\ &= \int_{[0,1]^2} c(u,v) \log c(u,v)\ dudv \\ &= E\Big(\log c(U,V)\Big). \end{aligned} \tag{3.1}$$

Blumentritt and Schmid [1] demonstrated the general form of the mutual information in (3.1) for Gaussian copula as

$$MI(X,Y) = -\frac{1}{2}\log(1-\rho^2).$$

Also, they provided mutual information for T copula as

$$\begin{aligned} MI(X,Y) = &-\frac{1}{2}\log(1-\rho^2) + 2\log\Big(\sqrt{\frac{\nu}{2\pi}}\ \beta(\frac{1}{2},\frac{\nu}{2})\Big) \\ &- \frac{2+\nu}{\nu} + (1+\nu)\Big[\psi(\frac{\nu+1}{2}) - \psi(\frac{\nu}{2})\Big], \end{aligned}$$

where $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta function and and $\psi(a) = \frac{\partial}{\partial a}\ln\Gamma(a)$ is the digamma function.

Let $(X_1,Y_1),(X_2,Y_2),...,(X_n,Y_n)$ be a random sample of size n from a pair (X,Y). We consider plug-in estimators of the copula-based mutual information as

$$\widehat{MI}(X,Y) = \frac{1}{n}\sum_{i=1}^{n} \log \widehat{c}_n(\tilde{U}_i, \tilde{V}_i). \tag{3.2}$$

To estimate the copula density $\widehat{c}_n$ in (3.2), we use the $\mathcal{LLPT}$ method. We call this estimator semiparametric because we use pseudo observations in the estimation of U and V as $\tilde{U}_i = n\hat{F}_X(x_i)/(n+1)$, $\tilde{V}_i = n\hat{F}_Y(y_i)/(n+1)$ for $i = 1,\cdots,n$ where $\hat{F}_X$ and $\hat{F}_Y$ are the empirical cumulative distribution function of the observation X_i and Y_i, respectively.

4 Simulation study

A simulation study is performed to evaluate the finite sample properties of the suggested estimator for copula based mutual information. We consider the beta kernel and Bernstein methods in [1] to compare with the $\mathcal{LLPT}$ method for copula density estimation. The data are generated from Gaussian and T ($\nu = 2, 10, 50$) copulas with different Kendall's tau ($\tau = 0.2, 0.5, 0.8$). The accuracy of estimators according to (3.2) are compared by 1000 Monte Carlo samples in terms of bias and root of mean square error (RMSE). The bias and RMSE of estimators are presented in Table 1 and 2 for sample sizes 50 and 200, respectively.

Simulation results show that estimated Bias and RMSE of mutual information decrease as sample size increases. By increasing Kendall's tau, Bias of mutual information decrease and RMSE of mutual information increases. Errors do not have a monotone behavior with respect to the degree of freedom in T copula. In general, the $\mathcal{LLPT}$ method has the best performance compared to the beta kernel and Bernstein methods in estimating mutual information. It can also be seen that the accuracy of the Bernstein method is better than the beta kernel method.

Table 1: Bias and RMSE of estimators for sample size 50

Copula	Kendall's tau	Beta kernel		Bernstein		LLPT	
		Bias	RMSE	Bias	RMSE	Bias	RMSE
Gaussian	0.2	0.1105	0.0204	0.1064	0.0159	0.1018	0.0152
	0.5	0.0901	0.0275	0.0845	0.0238	0.0836	0.0219
	0.8	0.0890	0.0276	0.0837	0.0266	0.0798	0.0253
T($\nu = 2$)	0.2	0.1069	0.0220	0.1031	0.0208	0.1023	0.0177
	0.5	0.0995	0.0334	0.0949	0.0322	0.0933	0.0290
	0.8	0.0914	0.0369	0.0872	0.0345	0.0858	0.0328
T($\nu = 10$)	0.2	0.1153	0.0231	0.1096	0.0202	0.1075	0.0165
	0.5	0.1010	0.0306	0.0971	0.0305	0.0924	0.0249
	0.8	0.0929	0.0379	0.0869	0.0350	0.0847	0.0291
T($\nu = 50$)	0.2	0.1059	0.0225	0.1035	0.0205	0.1034	0.0193
	0.5	0.0936	0.0294	0.0884	0.0303	0.0857	0.0253
	0.8	0.0906	0.0335	0.0864	0.0311	0.0832	0.0284

5 Conclusion

In this paper, a semiparametric estimation of copula-based mutual information using the $\mathcal{LLPT}$ method was suggested. This measure for Gaussian and T copulas on elliptical copula class was provided. The simulation results showed that the suggested estimator outperforms than beta kernel and Bernstein methods.

Table 2: Bias and RMSE of estimators for sample size 200

Copula	Kendall's tau	Beta kernel		Bernstein		LLPT	
		Bias	RMSE	Bias	RMSE	Bias	RMSE
Gaussian	0.2	0.0647	0.0120	0.0629	0.0094	0.0584	0.0052
	0.5	0.0543	0.0161	0.0497	0.0134	0.0466	0.0092
	0.8	0.0495	0.0178	0.0467	0.0158	0.0441	0.0142
T($\nu = 2$)	0.2	0.0593	0.0100	0.0571	0.0084	0.0539	0.0060
	0.5	0.0528	0.0155	0.0504	0.0149	0.0456	0.0128
	0.8	0.0356	0.0236	0.0322	0.0191	0.0272	0.0155
T($\nu = 10$)	0.2	0.0621	0.0130	0.0621	0.0093	0.0603	0.0058
	0.5	0.0565	0.0190	0.0543	0.0148	0.0502	0.0106
	0.8	0.0487	0.0200	0.0468	0.0172	0.0452	0.0132
T($\nu = 50$)	0.2	0.0701	0.0112	0.0670	0.0105	0.0633	0.0070
	0.5	0.0558	0.0178	0.0521	0.0131	0.0477	0.0121
	0.8	0.0471	0.0250	0.0460	0.0225	0.0443	0.0186

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FGM Copula based Analysis of Wireless Communication Performances for Multi-User Channels

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Abstract

The dependence of fading coefficients of wireless communication channels on each other affects communication performances such as Outage probability (OP), coverage region, energy efficiency, and secrecy capacity, possibly being constructive or destructive. In this paper, the outage probability as one of the most important wireless communication performances is investigated by using Copula theory. For this purpose, a wireless three-user multiple access channel (MAC) with Rayleigh fading and independent sources is considered and the outage probability in positive and negative dependence cases is compared. The results show that a negative dependence structure reduces the outage probability (compared to the independent state), but a positive dependence structure increases it.

Keywords: multiple access channel , Rayleigh fading, dependence, Copula theory, outage probability

1 Introduction

According to the surveys, there are more than four billion wireless subscribers worldwide, and the demand for wireless communication services is increasing daily, so we need to improve the methods

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of using available spectrum sources. One way to increase spectral efficiency is to use multi-user and multi-antenna communication systems in wireless communications. Disturbances caused by radio propagation environments always affect the performance of wireless communication systems. Fading is one of the disturbances of wireless propagation environments. In a wireless fading channel, the channel coefficients are interdependent random variables for which different distributions have been proposed so far[2, 19, 23]. In recent years, there have been many studies about fading channels; in many of these studies, the fading coefficients of wireless communication channels have been assumed to be independent of each other.

It is essential to investigate the performance of wireless communication systems under the influence of the dependence of channel fading coefficients.

Copula theory can be used as a valuable and powerful tool for modeling the dependence between random variables. This theory was first introduced by Sklar in 1959 [21]. Copulas are functions that relate the multivariate distribution function to its marginal distribution functions. Copula theory is widely used in statistics, economics, image processing, machine learning, Internet of Things (IoT), and engineering [4, 16, 18, 20, 24]. Also, recently, Copula theory is used to evaluate the effect of dependence of wireless channel coefficients on wireless communication performances, including outage probability, coverage region, energy efficiency, and secrecy capacity[8, 9, 17]. In [8], by using Copula theory, the effect of the dependence between the Rayleigh fading channel coefficients on the outage probability and the coverage region, two important communication performances, has been evaluated. In [9], the investigated channel is a doubly dirty fading MAC with non-causally known side information at transmitters; closed-form expressions for the outage probability and the coverage region have been obtained using the Copula theory. In [17], the channel coefficients have been considered interdependent, and a general closed-form expression has been obtained for the outage probability assuming an arbitrary fading distribution.

The Farlie-Gumbel-Morgenstern (FGM) Copulas, first studied by Eyraud, Farlie, Gumble, and Morgenstern [5, 6, 10, 15], are a well-known family of Copulas and have many properties [3, 7, 11, 12, 13]. This family of Copulas has a simple form, and the dependence parameter of these Copulas includes positive and negative values and zero. Also, FGM Copulas are the simplest to calculate joint distributions. Due to these properties, these Copulas are suitable for analyzing wireless channels with dependent coefficients.

In this paper, we study a wireless three-user fading MAC with independent sources and coherent receiver (the receiver knows the channel coefficients). We consider the channel coefficients to be dependent on each other to investigate the effect of the dependence of channel coefficients on wireless communication performances. First, we obtain a closed-form expression for the outage probability using the FGM Copula; then, according to this closed-form expression, we investigate the effect of the dependence of the channel coefficients on the outage probability. To this end, we compare the outage probability in dependent and independent cases and evaluate the impact of positive and negative dependencies on the outage probability.

The structure of this paper is as follows: Copula theory is described in sections 2. Communication channel is described in sections 3. The outage probability is obtained in Section 4. Numerical results are in section 5 and the paper is concluded in section 6.

2 Copula Theory

In this section, we briefly review some definitions and theorems of the Coppola theory that are used in the following sections [16].

Definition 1. *A d-dimensional Copula is a functin* $C : [0,1]^d \to [0,1]$ *subject to:*

- *C is a grounded function, that is:*

$$C(u_1,\dots,u_d) = 0; \ \textit{if any } u_j = 0, \ j \in \{1,\dots,d\}$$

- *The marginals of C are uniform, that is:*

$$C(1,\dots,1,u_j,1,\dots,1) = u_j; \ \forall \ j \in \{1,\dots,d\}$$

- *C is d-increasing on* $[0,1]^d$ *, that is:*

$$\sum_{i_1=1}^{2} \dots \sum_{i_d=1}^{2} (-1)^{i_1+\dots+i_d} C(u_{1i_1},\dots,u_{di_d}) \geq 0 \quad \textit{for all} \quad 0 \leq u_{j1} \leq u_{j2} \leq 1 \quad \textit{and } j \in \{1,\dots,d\}.$$

Theorem 2.1. *Suppose F is a multivariate joint cumulative distribution function (CDF) with marginals* $F_1,\dots,F_d$ *then there exists a Copula, C, such that:*

$$F(x_1,\dots,x_d) = C(F_1(x_1),\dots,F_d(x_d)) \tag{2.1}$$

If $F_i; \forall i \in \{1,\dots,d\}$ *is continuous, then C is unique, otherwise C is uniquely determined only on* $Ran(F_1) \times \dots \times Ran(F_d)$*. Conversely, consider a Copula, C, and univariate CDF's* $F_1,\dots,F_d$*, Then F defined in (2.1) is a multivariate CDF with marginals* $F_1,\dots,F_d$*.*

Corollary 1. *The joint probability density function (PDF) corresponding to* $F(x_1,\dots,x_d)$ *is:*

$$f(x_1,\dots,x_d) = f_1(x_1)\dots f_d(x_d)\, c(F_1(x_1),\dots,F_d(x_d)) \tag{2.2}$$

Where $f_i(x_i); i \in \{1,\dots,d\}$ *are the marginal PDFs of* $f(x_1,\dots,x_d)$ *and c is the Copula density function.*

The density function of Copula $C(u_1,\dots,u_d)$ *is given as:*

$$c(u_1,\dots,u_d) = \frac{\partial^d C(u_1,\dots,u_d)}{\partial u_1 \dots \partial u_d} \tag{2.3}$$

Definition 2. *A d-dimensional FGM Copula is defined as [13]:*

$$C(u_1,\dots,u_d) = \left(\prod_{j=1}^{d} u_j\right)\left(1 + \sum_{k=2}^{d} \sum_{1\leq j_1<\dots<j_k\leq d} \theta_{j_1\dots j_k} \overline{u}_{j1}\dots\overline{u}_{jk}\right)$$

$$\textit{Where } (u_1,\dots,u_d) \in [0,1]^d \textit{ and } \overline{u}_j = 1-u_j, \ j \in \{1,\dots,d\} \tag{2.4}$$

3 Communication Channel

We study a wireless three-user fading MAC with independent sources and coherent receiver that the channel coefficients are dependent on each other (Figure 1).

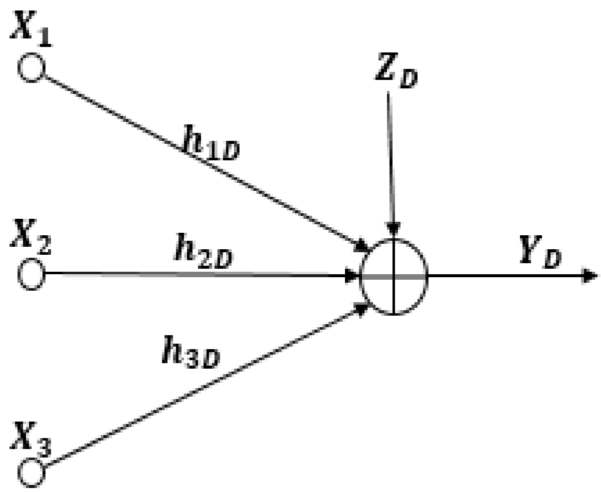

Figure 1: A three-user wireless Rayleigh fading MAC

The received signal is:

$$Y_D = h_{1D}X_1 + h_{2D}X_2 + h_{3D}X_3 + Z_D \tag{3.1}$$

Where the signals sent by the first, second, and third transmitters are X_1, X_2 and X_3, respectively.

$h_{iD}; i\in \{1,2,3\}$ are the fading coefficients of the channel between the transmitter i and the receiver. Z_D is independent identically distributed (i.i.d) Additive White Gaussian Noise (AWGN) with zero mean and variance N.

The capacity region of a three-transmitter wireless MAC with block fading and coherent receiver is (extension of the capacity region of two-user MAC with independent sources [1, 14]):

$$
\begin{aligned}
R_1 &\leq \frac{1}{2}\log_2\left(1+\frac{P_1\left|h_1\right|^2}{N}\right)\\
R_2 &\leq \frac{1}{2}\log_2\left(1+\frac{P_2\left|h_2\right|^2}{N}\right)\\
R_3 &\leq \frac{1}{2}\log_2\left(1+\frac{P_3\left|h_3\right|^2}{N}\right)\\
R_1+R_2 &\leq \frac{1}{2}\log_2\left(1+\frac{P_1\left|h_1\right|^2+P_2\left|h_2\right|^2}{N}\right)\\
R_1+R_3 &\leq \frac{1}{2}\log_2\left(1+\frac{P_1\left|h_1\right|^2+P_3\left|h_3\right|^2}{N}\right)\\
R_2+R_3 &\leq \frac{1}{2}\log_2\left(1+\frac{P_2\left|h_2\right|^2+P_3\left|h_3\right|^2}{N}\right)\\
R_1+R_2+R_3 &\leq \frac{1}{2}\log_2\left(1+\frac{P_1\left|h_1\right|^2+P_2\left|h_2\right|^2+P_3\left|h_3\right|^2}{N}\right)
\end{aligned}
\tag{3.2}
$$

Where R_1, R_2 and R_3 are the desired transmission rates of the first, second, and third transmitters, respectively.

4 Outage Probability

The outage probability of a three-user wireless Rayleigh correlated fading MAC is:

$$P_{out} = 1 - P^c \tag{4.1}$$

$$
\begin{aligned}
P^c =& \mathcal{A}_1 + \theta_{12}\left(\mathcal{A}_1 - 2\mathcal{A}_2 - 2\mathcal{A}_3 + 4\mathcal{A}_4\right)\\
&+ \theta_{13}\left(\mathcal{A}_1 - 2\mathcal{B}_1 - 2\mathcal{A}_3 + 4\mathcal{B}_3\right)\\
&+ \theta_{23}\left(\mathcal{A}_1 - 2\mathcal{B}_1 - 2\mathcal{A}_2 + 4\mathcal{B}_2\right)\\
&+ \theta_{123}\left(\mathcal{A}_1 - 2\mathcal{A}_2 - 2\mathcal{A}_3 + 4\mathcal{A}_4 - 2\mathcal{B}_1 + 4\mathcal{B}_2 + 4\mathcal{B}_3 - 8\mathcal{B}_4\right)
\end{aligned}
\tag{4.2}
$$

Where in the equation (4.2) we have:

$$\mathcal{A}_1 = \frac{\left(\overline{\nu}_1\right)^2 e^{\frac{-\mathcal{T}}{\overline{\nu}_1}}}{\left(\overline{\nu}_1 - \overline{\nu}_2\right)\left(\overline{\nu}_1 - \overline{\nu}_3\right)} \tag{4.3}$$

$$\mathcal{A}_2 = \frac{\left(\overline{\nu}_1\right)^2 e^{\frac{-\mathcal{T}}{\overline{\nu}_1}}}{\left(2\overline{\nu}_1 - \overline{\nu}_2\right)\left(\overline{\nu}_1 - \overline{\nu}_3\right)} \tag{4.4}$$

$$\mathcal{A}_3 = \frac{(\overline{\nu}_1)^2 e^{\frac{-2\mathcal{T}}{\overline{\nu}_1}}}{2(\overline{\nu}_1 - 2\overline{\nu}_2)(\overline{\nu}_1 - 2\overline{\nu}_3)} \tag{4.5}$$

$$\mathcal{A}_4 = \frac{(\overline{\nu}_1)^2 e^{\frac{-2\mathcal{T}}{\overline{\nu}_1}}}{4(\overline{\nu}_1 - \overline{\nu}_2)(\overline{\nu}_1 - 2\overline{\nu}_3)} \tag{4.6}$$

$$\mathcal{B}_1 = \frac{(\overline{\nu}_1)^2 e^{\frac{-\mathcal{T}}{\overline{\nu}_1}}}{(\overline{\nu}_1 - \overline{\nu}_2)(2\overline{\nu}_1 - \overline{\nu}_3)} \tag{4.7}$$

$$\mathcal{B}_2 = \frac{(\overline{\nu}_1)^2 e^{\frac{-\mathcal{T}}{\overline{\nu}_1}}}{(2\overline{\nu}_1 - \overline{\nu}_2)(2\overline{\nu}_1 - \overline{\nu}_3)} \tag{4.8}$$

$$\mathcal{B}_3 = \frac{(\overline{\nu}_1)^2 e^{\frac{-2\mathcal{T}}{\overline{\nu}_1}}}{4(\overline{\nu}_1 - 2\overline{\nu}_2)(\overline{\nu}_1 - \overline{\nu}_3)} \tag{4.9}$$

$$\mathcal{B}_4 = \frac{(\overline{\nu}_1)^2 e^{\frac{-2\mathcal{T}}{\overline{\nu}_1}}}{8(\overline{\nu}_1 - \overline{\nu}_2)(\overline{\nu}_1 - \overline{\nu}_3)} \tag{4.10}$$

In the above equations, $\overline{\nu}_1$, $\overline{\nu}_2$, and $\overline{\nu}_3$ are the average signal-to-noise ratios (SNR) at transmitters t_1, t_2, and t_3, respectively and $\mathcal{T} = 2^{2R_0} - 1$ that R_0 represents the total required threshold information rates.

Proof. Outage probability is the probability that the information rate is greater than the random capacity of the channel or less than a required threshold information rate. According to this definition and considering that any of the inequalities in equation (3.2) can be used to calculate the outage probability, we have:

$$\begin{aligned} P_{out} &= Pr\left(R_1 + R_2 + R_3 \le R_0\right) && (4.11)\\ &= 1 - Pr\left(R_1 + R_2 + R_3 > R_0\right) && (4.12)\\ &= 1 - P^c && (4.13) \end{aligned}$$

Where P^c is the complement of the outage probability.

$$\begin{aligned} P^c &= Pr\left(\frac{1}{2}\log_2\left(1 + \frac{P_1|h_1|^2 + P_2|h_2|^2 + P_3|h_3|^2}{N}\right) > R_0\right) && (4.14)\\ &= Pr\left(\frac{P_1|h_1|^2 + P_2|h_2|^2 + P_3|h_3|^2}{N} > 2^{2R_0} - 1\right) && (4.15)\\ &= Pr\left(\nu_1 + \nu_2 + \nu_3 > \mathcal{T}\right) && (4.16)\\ &= \int_0^\infty \int_0^\infty \int_{\mathcal{T}-\nu_2-\nu_3}^\infty f\left(\nu_1, \nu_2, \nu_3\right) d\nu_1 d\nu_2 d\nu_3 && (4.17) \end{aligned}$$

Where ν_1, ν_2 and ν_3 are SNRs at transmitters t_1, t_2, and t_3, respectively and we have:

$$\nu_i = \frac{P_i|h_i|^2}{N};\ i \in \{1, 2, 3\} \tag{4.18}$$

$f(\nu_1,\nu_2,\nu_3)$ in equation (4.17) is the joint PDF of ν_1, ν_2 and ν_3, and we can calculate it according to equation (2.2)

$$f(\nu_1,\nu_2,\nu_3)=f_1(\nu_1)\,f_2(\nu_2)\,f_3(\nu_3)\,c\left(F_1(\nu_1),F_2(\nu_2),F_3(x_3)\right) \tag{4.19}$$

Where $f(\nu_i);i\in\{1,2,3\}$ and $F(\nu_i);i\in\{1,2,3\}$ are PDFs and CDFs of $\nu_i;i\in\{1,2,3\}$, respectively and $c\left(F_1(\nu_1),F_2(\nu_2),F_3(x_3)\right)$ is the density function of three-dimensional FGM Copula.

The channel coefficients, $h_i;i\in\{1,2,3\}$, have a Rayleigh distribution, so $|h_i|^2;i\in\{1,2,3\}$ and consequently $\nu_i;i\in\{1,2,3\}$ have an exponential distribution.

$$f(\nu_i)=\frac{1}{\overline{\nu}_i}\exp\left(-\frac{\nu_i}{\overline{\nu}_i}\right);i\in\{1,2,3\} \tag{4.20}$$

$$F(\nu_i)=1-\exp\left(-\frac{\nu_i}{\overline{\nu}_i}\right);i\in\{1,2,3\} \tag{4.21}$$

Now we calculate $c\left(F_1(\nu_1),F_2(\nu_2),F_3(x_3)\right)$. Considering $d=3$ in equation (2.4), three-dimensional FGM Copula is obtained as:

$$C(u_1,u_2,u_3)=u_1u_2u_3\left(1+\theta_{12}\overline{u}_1\overline{u}_2+\theta_{13}\overline{u}_1\overline{u}_3+\theta_{23}\overline{u}_2\overline{u}_3+\theta_{123}\overline{u}_1\overline{u}_2\overline{u}_3\right) \tag{4.22}$$

Where $\overline{u}_i=1-u_i;i\in\{1,2,3\}$ and θ_{12}, θ_{13}, θ_{23} and θ_{123} are the FGM Copula parameters. According to (2.3) and (4.22), the density function of the three-dimensional FGM Copula is:

$$\begin{aligned}c(u_1,u_2,u_3)=1&+\theta_{12}(1-2u_1)(1-2u_2)\\&+\theta_{13}(1-2u_1)(1-2u_3)\\&+\theta_{23}(1-2u_2)(1-2u_3)\\&+\theta_{123}(1-2u_1)(1-2u_2)(1-2u_3)\end{aligned} \tag{4.23}$$

Now according to (4.19), (4.20) and (4.23), $f(\nu_1,\nu_2,\nu_3)$ is obtained as follows:

$$\begin{aligned}f(\nu_1,\nu_2,\nu_3)=\frac{e^{-\frac{\nu_1}{\overline{\nu}_1}-\frac{\nu_2}{\overline{\nu}_2}-\frac{\nu_3}{\overline{\nu}_3}}}{\overline{\nu}_1\overline{\nu}_2\overline{\nu}_3}\Bigg[&1+\theta_{12}\left(1-2e^{-\frac{\nu_1}{\overline{\nu}_1}}\right)\left(1-2e^{-\frac{\nu_2}{\overline{\nu}_2}}\right)\\&+\theta_{13}\left(1-2e^{-\frac{\nu_1}{\overline{\nu}_1}}\right)\left(1-2e^{-\frac{\nu_3}{\overline{\nu}_3}}\right)\\&+\theta_{23}\left(1-2e^{-\frac{\nu_2}{\overline{\nu}_2}}\right)\left(1-2e^{-\frac{\nu_3}{\overline{\nu}_3}}\right)\\&+\theta_{123}\left(1-2e^{-\frac{\nu_1}{\overline{\nu}_1}}\right)\left(1-2e^{-\frac{\nu_2}{\overline{\nu}_2}}\right)\left(1-2e^{-\frac{\nu_3}{\overline{\nu}_3}}\right)\Bigg]\end{aligned} \tag{4.24}$$

By putting (4.22) in (4.17), it is easy to calculate the triple integral and the outage probability is obtained as (4.2)-(4.10) and the proof is complete.

5 Numerical Results

Numerical results are presented in this section. According to these results, we can investigate the effect of positive and negative dependencies on the outage probability performance.

In Figure 2, the outage probability is plotted in terms of average SNR. According to this figure, as the SNR increases, the channel condition improves, so the outage probability decreases. Also, we see that the negative dependence structure reduces the outage probability compared to the independent case, which means an improvement in the outage probability performance.

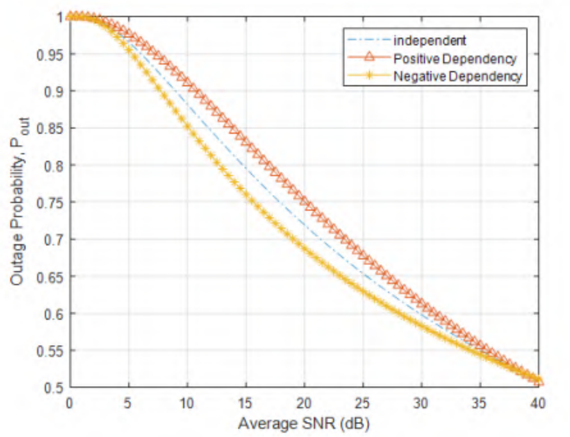

Figure 2: Outage probability versus average SNR

Conversely, we see that the positive dependence structure increases the outage probability compared to the independent case, that is, positive dependence has a detrimental effect on the outage probability performance.

6 Conclusion

In this paper, wireless three-user MAC with independent sources and Rayleigh fading was investigated. Using the FGM Copula, a closed form expression for the outage probability was obtained. Then we analyzed the impact of positive and negative dependencies on the outage probability performance. According to the obtained results, it is clear that negative dependence, compared to the independent state, reduces the outage probability, while positive dependency increases the outage probability compared to the non-dependent case.

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Ordering of Conditional Asymmetry

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Abstract

The aim of this paper is to study ordering of conditional asymmetry for copulas like those resulting from the concordance ordering for dependence. Several example provided to illustrate the result.
Keywords: Bivariate symmetry, Conditional asymmetry, Copula, Ordering.

1 Introduction

Let (X, Y) be a pair of continuous random variables with the joint distribution function $F(x, y) = P(X \leq x, Y \leq y)$, and univariate marginal distributions $F_1(x) = P(X \leq x)$, $F_2(y) = P(Y \leq y)$, at each $x, y \in \mathbb{R}$. Let $F(y|x) = P(Y \leq y|X = x)$ denote the conditional distribution of Y given $X = x$. Formally, the random variable Y is conditionally symmetric given $X = x$ if $Y|X = x$ is symmetric; i.e., $F(y|x) = 1 - F(-y|x)$. In some regression and time series models, some distributional assumptions are often imposed on the error term. One of these assumptions is the conditional symmetry around zero of error term given the independent variables [2]. The assumption of conditional symmetry is commonly used in adaptive estimation. Bickel [3] showed that under conditional symmetry of error terms, adaptive estimation of the parameters in regression model

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achieve the same information bound as the maximum likelihood estimator. The existence or lack of conditional symmetry is also important in modelling time series financial data. For example, the symmetry of the residuals implies that positive forecasts errors to the conditional mean are as likely as negative forecasts errors. If this is not the case, the forecasts should adjust to the possibility that positive and negative forecasts errors are not equally likely [2]. But conditional symmetry is equivalent to $(X,Y) \stackrel{d}{=} (X,-Y)$, where $\stackrel{d}{=}$ denotes equality in distributions or $F_{X,Y}(x,y) = F_{X,-Y}(x,y)$, for all x, y. Following Sklar's Theorem ([5]) the joint cumulative distribution function of X and Y can then be expressed, at each $x, y \in R$ as $F(x,y) = C\{F_1(x), F_2(y)\}$ in terms of a unique copula C. Since many dependency characteristics of random variables can be studied by using copulas, they play an important role in modeling the dependence structure between random variables [4]. In practice, modelling the dependence structure of random variables, leads to the problem of fitting an appropriate copula to a given set of data. Many families of copulas with different dependence properties have been introduced. However, most of these copulas are unable to incorporate a feature of the data such as conditional symmetry and, hence, are not suitable for applications. For instance, if the error terms of a time series model has some degree of conditional asymmetry, then conditional symmetric models are not adequate. In view of Sklar's Theorem, the conditional symmetry property of random variables can also be studied in terms of their associated copula. The objective of the present work is to explore the possibility of ordering copulas by means of their asymmetry level and study its implications. This is motivated by providing a partial answer to this question: whether there are inequalities for conditional asymmetry of copulas, like those resulting from the concordance ordering for dependence?

2 Proposed Asymmetry Ordering

By using transformation $F_1(x) = u$ and $F_2(y) = v$, in view of Sklar's Theorem, under the symmetry of Y, i.e., $F_2(y) = 1 - F_2(-y)$ we have

$$P(Y \le y|X = x) = P(V \le v|U = u), \text{ and } P(-Y \le y|X = x) = P(1 - V \le v|U = u),$$

where $(U,V) = (F_1(X), F_2(Y))$ is a pair of uniform [0,1] random variables. Note that $P(V \le v|U = u) = \frac{\partial}{\partial u}C(u,v)$ and $P(1 - V \le v|U = u) = 1 - \frac{\partial}{\partial u}C(u, 1 - v)$; see, e.g., [5]. Now it is immediate that, if $Y \stackrel{d}{=} -Y$, then $(Y|X = x) \stackrel{d}{=} (-Y|X = x)$, if and only if $C(u,v) = u - C(u, 1 - v)$. We will use the notion $C^*(u,v) = u - C(u,1-v)$. We say that a copula C is conditionally symmetric if, $C(u,v) = C^*(u,v)$, for all $u, v \in [0.1]$. When a copula C is conditionally asymmetric, i.e., $C(u,v) \neq C^*(u,v)$, for some $u, v \in$[0,1]. For a given copula C, let $\hat{C}(u,v) = u+v-1+C(1-u,1-v)$ be the survival copula or reflected copula associated with C and $C^T(u,v) = P(U \le v, V \le u)$. A copula is radially symmetric if and only if $C(u,v) = \widehat{C}(u,v)$ and it is exchange symmetric if and only if $C(u,v) = C^T(u,v)$, for all $u, v \in [0,1]$.

For two pairs (X_1,Y_1) and (X_2,Y_2) with the associated copulas C_1 and C_2, the pair (X_1,Y_1) is said to be less dependent than the pair (X_2,Y_2), in the sense of concordance ordering, if $C_1(u,v) \le C_2(u,v)$ for all $(u,v) \in [0,1]^2$ [5]. Motivation by this definition, we provide a definition for conditional asymmetry.

Definition 1. *For two copulas C_1 and C_2, the copula C_2 is said to be more conditional asymmetric than the copula C_1 (denoted by $C_1 \prec_{CA} C_2$), if and only if, for all $(u,v) \in [0,1]^2$,*

$$| C_1(u,v) - C_1^*(u,v) | \leq | C_2(u,v) - C_2^*(u,v) |, \tag{2.1}$$

where $C_i^*(u,v) = u - C_i(u,1-v)$. We call the ordering $\prec_{CA}$ as the conditional asymmetry order for copulas.

For dependence concepts, by using the fact that for any copula C satisfies $W(u,v) \leq C(u,v) \leq M(u,v)$, for all $(u,v) \in [0,]^2$, where $W(u,v) = \max(u+v-1,0)$ and $M(u,v) = \min(u,v)$, the maximal and minimal members of the class of copulas are M and W, respectively. The copula M is the dependence structure of the compelete positive dependent random variables X and Y; i.e., when $P(Y = f(X)) = 1$, for a non-decreasing function f and W is the copula of complete negative dependent random variables X and Y; i.e., when $P(Y = f(X)) = 1$, for a non-increasing function f. The following result shows that for asymmetry concept it is not the case. Let $\mathcal{CA}$ denotes the class of conditionally asymmetric copulas.

Proposition 2.1. *The smallest member of $\mathcal{CA}$ with respect to order $\prec_{CA}$ does not exist.*

Proof. If C_1 be a given conditional symmetric copula, then for any copula C_2 we have that

$$0 =| C_1(u,v) - {C^*}_1(u,v) | \leq | C_2(u,v) - {C^*}_2(u,v) |,$$

that is $C_1 \prec_{CA} C_2$. If C_3 is the smallest member of $\mathcal{CA}$, then $C_3 \prec_{CA} C_1$, i.e., $0 < C_3(u,v) - C_3^*(u,v) | \leq | C_1(u,v) - C_1^*(u,v) |= 0$, for all $u,v \in [0,1]$ and thus $C_3(u,v) = C_3^*(u,v)$, which means that the minimal member of $\mathcal{CA}$ must be a conditional symmetric copula. □

Theorem 2.2. *The copulas M and W are the greatest members $\mathcal{CA}$ with respect to $\prec_{CA}$.*

Proof. For any copula C we have

$$| C(u,v) - C^*(u,v) \leq \min(u,v,1-u,1-v),$$

for all $(u,v) \in [0,1]^2$. Thus the the greatest element of $\mathcal{CA}$ with respect to $\prec_{CA}$ is the copula D satisfying $| D(u,v) - D^*(u,v) |= \min(u,v,1-u,1-v)$, for all $(u,v) \in [0,1]^2$. Since $M^* = W$, $W^* = M$ and $| W(u,v) - M(u,v) |= M(u,v) - W(u,v) = \min(u,v,1-u,1-v)$ for all $(u,v) \in [0,1]^2$, then we must have $D = M$ or $D = W$. □

Example 2.3. Let $C(u,v) = uv[1+\theta(1-u)(1-v)]$, $u,v \in [0,1]$ and $\theta \in [-1,1]$, be the FGM copula. Then $| C(u,v) - C^*(u,v) |= 2uv(1-u)(1-v)|\theta|$. For $v \leq u$ and $v \leq 1-u$, $\min(u,v,1-u,1-v) = v$. If $u \leq \frac{1}{2}$ then $2uv(1-u)(1-v) | \theta | \leq v(1-u)(1-v) | \theta | \leq v$ and if $u \geq \frac{1}{2}$ then $2(1-u) \leq 1$ and $2uv(1-u)(1-v) | \theta | \leq uv(1-v) | \theta | \leq v$. Therefore $2uv(1-u)(1-v) | \theta | \leq \min(u,v,1-u,1-v)$ and thus $C \prec_{CA} M$ and $C \prec_{CA} W$.

The following results provides an invariance property for $\prec_{CA}$ order.

Proposition 2.4. *If C_1 and C_2 be two copulas such that $C_1 \prec_{CA} C_2$ then $C_1^* \prec_{CA} C_2^*$, $C_1^T \prec_{CA} C_2^T$, $\hat{C}_1 \prec_{CA} \hat{C}_2$.*

Proof. The ordering $C_1^* \prec_{AC} C_2^*$ follows directly from definition 2.1. Since $\mid C^T(u,v) - C^{T^*}(u,v) \mid = \mid C(v,u) - v + C(v,1-u) \mid$, letting $1-u=t$, we have that $\mid C(v,1-t) - v + C(v,t) \mid = \mid C(v,t) - C^*(v,t) \mid$ and thus $C_1^T \prec_{CA} C_2^T$. Similary, for $|\hat{C} - \hat{C}^*| = |C(1-u,1-v) - (1-u-C(1-u,v))|$, letting $1-u=t$ and $1-v=s$, we have that $\mid C(t,s) - (t - C(t,1-s)) \mid = \mid C(t,s) - C^*(t,s) \mid$, and then $\hat{C}_1 \prec_{CA} \hat{C}_2$. □

3 Some Examples

In this section we provide several examples for proposed $\prec_{AC}$ order.

Example 3.1. Let $C_{\alpha,\beta}$ be the two-parameter family of copulas given by [5]

$$C_{\alpha,\beta}(u,v) = uv + uv(1-u)(1-v)(\alpha + (\beta - \alpha)v(1-u)) \tag{3.1}$$

where $\alpha \in [-1,1]$ and $(\alpha - 3 - \sqrt{9 + 6\alpha - 3\alpha^2})/2 \le \beta \le 1$. Let $D_{\alpha,\beta}(u,v) = \frac{C_{\alpha,\beta}(u,v) + C^{**}_{\alpha,\beta}(u,v)}{2}$. Since $\mid D_{\alpha,\beta}(u,v) - D^*_{\alpha,\beta}(u,v) \mid = uv(1-u)(1-v) \mid 1-2u \mid\mid \beta - \alpha \mid$, by using (2.1), it follows that the family of copulas $\{D_{\alpha,\beta}\}$, is ordered with respect to $\prec_{CA}$ if, and only if, for a fixed α and any β_1 and β_2, $\mid \beta_1 - \alpha \mid \le \mid \beta_2 - \alpha \mid$. Similarly, for a fixed β and any α_1 and α_2, this family of copulas is ordered with respect to $\prec_{CA}$ if, and only if, $\mid \beta - \alpha_1 \mid \le \mid \beta - \alpha_2 \mid$.

Example 3.2. Let $K(u,v) = uv + uv(1-u)(1-v)(\frac{1}{2} - u)$ and $D_{\alpha,\beta}$ be the copula given by (3.1). Then it is easy to check that $K \prec_{CA} D_{\alpha,\beta}$ if and only if, $\mid \beta - \alpha \mid \ge 1$.

Example 3.3. For two copulas C_0 and C_1, let $C_\theta(u,v) = \theta C_1(u,v) + (1-\theta)C_0(u,v)$, $\theta \in [0,1]$, which is always a copula [5]. Then $C^*_\theta(u,v) = \theta C^*_1(u,v) + (1-\theta)C^*_0(u,v)$ and

$$|C_\theta(u,v) - C^*_\theta(u,v)| = |\theta(C_1(u,v) - C^*_1(u,v) + (1-\theta)(C_0(u,v) - C^*_0(u,v))|.$$

If C_0 conditionally symmetric, i.e., $C_0 = C_0^*$, then C_θ is positively ordered with respect to $\prec_{CA}$, that is for $\theta_1 \le \theta_2$, $C_{\theta_1} \prec_{CA} C_{\theta_2}$. If C_1 conditionally symmetric, i.e., $C_1 = C_1^*$, then C_θ is negatively ordered with respect to $\prec_{CA}$, that is for $\theta_1 \ge \theta_2$, $C_{\theta_1} \prec_{CA} C_{\theta_2}$.

Example 3.4. Let $C_\theta(u,v) = \theta M(u,v) + (1-\theta)W(u,v)$. Since $C^*_\theta(u,v) = \theta W(u,v) + (1-\theta)M(u,v)$ and $|C_\theta(u,v) - C^*_\theta(u,v)| = |2\theta - 1|(M(u,v) - W(u,v))$, then C_θ is positively (resp, negatively) ordered with respect to $\prec_{CA}$ for $\theta > \frac{1}{2}$ (resp, $\theta < \frac{1}{2}$). Note that for $\theta = \frac{1}{2}$ the copula C_θ is conditionally symmetric.

Example 3.5. Let C_θ be the copula defined by

$$C_\theta(u,v) = uv + \theta\phi(u)\phi(v), \tag{3.2}$$

where $\theta \in [-1,1]$ and ϕ is a function on $[0,1]$ with $\phi(0) = \phi(1) = 0$ and $|\phi(u) - \phi(v)| \le |u-v|$ for all $u,v \in [0,1]$ [1]. Note that if $\phi(u) + \phi(1-u) = 0$ for all $u \in [0,1]$, then C_θ is conditionally symmetric, otherwise C_θ is conditionally asymmetric. Since $|(C_\theta(u,v) - C^*_\theta(u,v)| = |\theta||\phi(u)[\phi(v) - \phi(1-v)]|$, then C_θ is positively (resp, negatively) ordered with respect to $\prec_{CA}$ for $\theta > 0$ (resp, $\theta < 0$).

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Sharpe Ratio Analysis: A Copula approach

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Abstract

Sharpe ratio is a commonly used risk-adjusted measure for evaluating portfolio performance in risk management. Despite of its popularity, whenever the returns are non-normal or dependent, the calculated Sharpe index is either over or under-estimated. The aim of this paper is to study the effect of dependence on the Sharpe ratio of a two assets portfolio by using the copula of its returns.

Keywords: Copula, Dependence, Portfolio, Risk-adjusted measure, Sharpe ratio

1 Introduction

Sharpe ratio [5, 4] is a common tool for comparing the performance of financial assets. Multivariate normal distribution is the commonly used model for analyzing return of financial assets. But normal distribution restricts the dependence of returns to be linear as measured by Pearson's correlation. In recent years, copulas are often used to measure the dependence between return of financial assets. Copulas separate the dependency between variables from their univariate marginal distributions. In this way, the dependence structure can be linear, nonlinear or tail dependent. Despite the use of copula approach to calculate many financial indices [1], little work has been done on the use of copulas for Sharp ratio estimation. In this paper, we study the effect of dependency on the Sharpe ratio using the copulas and compare it with the empirical method.

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2 Copula-based Sharpe ratio

Let R_A and R_B denote the returns of two assets A and B, respectively. Further, let R_f be the return of a benchmark investment strategy. Define continuous random variables of X and Y as $X = R_A - R_f$ and $Y = R_B - R_f$. Consider univariate marginal distribution functions of X and Y as $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$ for $x, y \in \mathbb{R}$ with the joint distribution function $H(x, y) = P(X \leq x, Y \leq y)$. Let $T = wX + (1-w)Y$, be a portfolio with dependent components X and Y, where $0 < w < 1$ is the weight of X and $(1-w)$ is the weight of Y. The Sharpe ratio of T is given by [5]

$$\mathrm{SR}_T = \frac{w\mu_X + (1-w)\mu_Y}{\sqrt{w^2\sigma_X^2 + (1-w)^2\sigma_Y^2 + 2w(1-w)\sigma_{X,Y}}}, \tag{2.1}$$

where $\mu_X = E(X)$, $\mu_Y = E(Y)$, $\sigma_X^2 = \mathrm{var}(X)$, $\sigma_Y^2 = \mathrm{var}(Y)$ and $\sigma_{X,Y} = \mathrm{cov}(X, Y)$. In this formula, μ_X, μ_Y, σ_X^2 and σ_Y^2 are calculated from the marginal distributions and $\sigma_{X,Y}$ is associated to the joint distribution function of X and Y. In view of the *Sklar's Theorem* [9], let C be the unique copula of the pair (X, Y) through the relation

$$H(x, y) = C(F(x), G(y)), \quad x, y \in \mathbb{R}.$$

In fact, C is the joint distribution function of the pair $(U, V) = (F(X), G(Y))$ of uniform (0,1) random variables.

By using the Hoeffding's identity[2] and transformations $u = F(x)$ and $v = G(y)$, we have

$$\begin{aligned}
\sigma_{X,Y} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[H(x,y) - F(x)G(y)]dxdy \\
&= \int_0^1\int_0^1 [C(u,v) - uv]dF^{-1}(u)dG^{-1}(v) \\
&= \int_0^1\int_0^1 F^{-1}(u)G^{-1}(v)dC(u,v) - \mu_X\mu_Y.
\end{aligned}$$

Let Π denote the copula of independent random variables, i.e., $\Pi(u, v) = uv$ for all $(u, v) \in [0, 1]^2$, and let M and W denote the Fréchet-Hoeffding upper and lower bound copulas, respectively, which, for any copula C, satisfy: $\max(u + v - 1, 0) = W(u, v) \leq C(u, v) \leq M(u, v) = \min(u, v)$ for every $(u, v) \in [0, 1]^2$. We recall that M (W) is the copula of perfect positive (negative) dependence random variables [9]. In the following example we compute the Sharpe ratio of a portfolio with two perfect dependence assets, having exponential distribution.

Example 2.1. Let X and Y be two exponential random variables with the means $\dfrac{1}{\lambda_1}$ and $\dfrac{1}{\lambda_2}$. If the copula of X and Y is M, then $\sigma_{X,Y} = \frac{1}{\lambda_1\lambda_2}$ and the Sharpe ratio of portfolio $T = wX+(1-w)Y$ is then

$$SR_T = \frac{\lambda_1 + (\lambda_2 - \lambda_1)w}{|\lambda_1 + (\lambda_2 - \lambda_1)w|} = \begin{cases} +1 & if\ w \geq \frac{\lambda_1}{\lambda_1 - \lambda_2} \\ 0 & if\ w = \frac{\lambda_1}{\lambda_1 - \lambda_2} \\ -1 & if\ w < \frac{\lambda_1}{\lambda_1 - \lambda_2}. \end{cases}$$

If the copula of X and Y is W, then $\sigma_{X,Y} = (1-\frac{\pi^2}{6})\frac{1}{\lambda_1\lambda_2}$ and the Sharpe ratio of the portfolio T is given by

$$SR_T = \frac{\lambda_1 + (\lambda_2-\lambda_1)w}{\sqrt{(1-w)^2\lambda_2^2 + w^2\lambda_1^2 + (\frac{\pi^2}{3}-2)w(1-w)}}.$$

The maximum value of SR is $\frac{6}{\sqrt{36-3\pi^2}} = 2.3734$, which happens at $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

Example 2.2. Let X and Y be two exponential random variables with the means $\dfrac{1}{\lambda_1}$ and $\dfrac{1}{\lambda_2}$, respectively. Let their dependence structure be the FGM copula [9] given by

$$C_\theta(u,v) = uv[1+\theta(1-u)(1-v)], \;\; u,v \in [0,1],$$

where $-1 \le \theta \le 1$. Then by using (2.2) we have $\sigma_{X,Y} = \dfrac{\theta}{4\lambda_1\lambda_2}$ and the Sharpe ratio of the portfolio T with the exponential returns and FGM copula structure is given by

$$SR_T = \frac{\lambda_1 + (\lambda_2-\lambda_1)w}{\sqrt{\lambda_2^2 w^2 + \lambda_1^2(1-w)^2 + \frac{w(1-w)\theta\lambda_1\lambda_2}{2}}}. \tag{2.2}$$

As a function of the weight w, the maximum value of SR_T is given by $SR_T^* = 2\sqrt{\frac{2}{4+\theta}}$, which happens at $w^* = \frac{\lambda_1}{\lambda_1+\lambda_2}$. We note that SR^* is decreasing in θ and for $\theta \in [-1,1]$, $SR^* \in [1.264, 1.633]$. Since θ is the dependency parameter, the value of the Sharp ratio decreases as the dependence between components of portfolio increases.

3 Properties of copula-based Sharpe ratio

Let $T_{C_\theta} = wX + (1-w)Y$ be a two assets portfolio whose components (X,Y) has the one-parameter copula structure C_θ. In this section, we discuss some properties of the copula-based Sharpe ratio $SR(T_{C_\theta}) = \frac{\mu(T_{C_\theta})}{\sigma(T_{C_\theta})}$, where $\mu(T_{C_\theta}) = E(T_{C_\theta})$ and $\sigma(T_{C_\theta}) = \sqrt{var(T_{C_\theta})}$. First note that if the variance of returns does not depend on their expected values, then $SR(T_{C_\theta})$ is decreasing (increasing) in θ, as $\sigma(T_{C_\theta})$ is increasing (decreasing) in θ. The following result compares two portfolios with the common marginal distributions and different dependence structures.

Proposition 3.1. *For $i = 1,2$, let (X_i, Y_i) has the copula C_{θ_i}, $i = 1,2$, and $E(X_1) = E(X_2)$, $var(X_1) = var(X_2)$. Then $SR(T_{C_{\theta_1}}) \ge SR(T_{C_{\theta_2}})$, for $\theta_1 \le \theta_2$, whenever C_θ is a positively ordered family of copulas; that is $C_{\theta_1}(u,v) \le C_{\theta_2}(u,v)$, for all $u,v \in [0,1]$ and $\theta_1 \le \theta_2$.*

The following result provides a lower and upper bound for Sharpe ratio of a two assets portfolio consists of dependent returns.

Proposition 3.2. *Let $T_C = wX + (1-w)Y$ be a two assets portfolio whose components (X,Y) has the copula structure C. Then*

$$SR(T_M) \le SR(T_C) \le SR(T_W),$$

where, M and W are the Fréchet-Hoeffding upper and lower bound copulas,

Table 1: Values of the Sharpe ratio (SR) for a two assets portfolio with various copula structure and various univariate marginal distributions

Kendall's τ	Copula structure	θ	SR (Normal margins)	SR (Beta margins)	SR (Exponential margins)
-1	W	—	1.91	17.32	2.18
-0.9	Clayton	-0.94	1.86	15.20	2.07
	Frank	-38.28	1.88	16.32	2.16
	Normal	-0.98	1.89	16.67	2.17
-0.5	Clayton	-0.66	1.66	9.91	1.72
	Frank	-5.73	1.67	9.79	1.85
	Normal	-0.70	1.69	10.07	1.87
-0.3	Clayton	-0.46	1.55	8.17	1.56
	Frank	-2.91	1.54	7.89	1.64
	Normal	-0.45	1.56	8.00	1.66
0	Π	—	1.38	6.20	1.38
0.3	Clayton	0.85	1.25	5.12	1.24
	Frank	2.91	1.26	5.27	1.20
	Gumbel	1.42	1.25	5.28	1.13
	Normal	0.45	1.25	5.22	1.18
	Joe	1.77	1.25	5.34	1.10
0.5	Clayton	2.00	1.20	4.76	1.17
	Frank	5.73	1.20	4.88	1.12
	Gumbel	2.00	1.20	4.89	1.06
	Normal	0.70	1.20	4.83	1.09
	Joe	2.85	1.20	4.97	1.04
0.9	Clayton	18.00	1.15	4.48	1.03
	Frank	38.28	1.14	4.50	1.00
	Gumbel	10.00	1.15	4.49	1.00
	Normal	0.98	1.14	4.49	1.00
	Joe	18.73	1.14	4.52	1.00
+1	M	—	1.14	4.47	1.00

A copula C is said to be positive quadrant dependence (PQD) if for all $(u, v) \in [0, 1]^2$, $C(u, v) \geq uv$ and negative quadrant dependence (NQD) if $C(u, v) \leq uv$ [9]. The following result compares the Sharp ratio of a portfolio with dependent returns with the Sharp ratio of a portfolio consists of independent returns.

Proposition 3.3. *Let $T_C = wX + (1-w)Y$ be a two assets portfolio whose components (X, Y) has a copula structure C. If C is PQD, then $SR(T_C) \leq SR(T_\Pi)$. If C is NQD then $SR(T_\Pi) \leq SR(T_C)$.*

4 Numerial results

The copula-based Sharpe ratio cannot be written as a closed formula for many copulas. In the following, we calculate the Sharpe ratio of a two assets portfolio for some copulas using

Table 2: Estimated values of the Sharpe ratio for three portfolios

Portfolio	Weight	Independence	Emprical method	Copula method
Portfolio (1)	0.2	0.0118	-0.0088	0.0110
	0.4	0.0164	-0.0066	0.0144
	0.6	0.0200	-0.0036	0.0173
	0.8	0.0206	-0.0001	0.0189
Portfolio (2)	0.2	0.0235	0.0043	0.0200
	0.4	0.0424	0.0197	0.0320
	0.6	0.0572	0.0346	0.0436
	0.8	0.0630	0.0477	0.0539
Portfolio (3)	0.2	0.0369	0.0191	0.0344
	0.4	0.0548	0.0361	0.0487
	0.6	0.0644	0.0488	0.0582
	0.8	0.0654	0.0556	0.0622

the Monte Carlo simulation. We use the copulas Π, M, W, normal, Clayton, Frank, Gumbel and Joe. The level of dependence is fixed for each copula in terms of its Kendall's $\tau \in \{-1, -0.9, -0.5, -0.1, 0, 0.1, 0.5, 0.9\}$. By solving the equation $\tau(\theta) = \tau_n$ for θ, where τ_n is the empirical version of the Kendall's tau of the simulated data, the corresponding parameter of each copula is computed. For computing the Sharpe ratio of a two assets portfolio $T = wX + (1-w)Y$, we consider three cases for marginal distributions: Normal $(\frac{5}{7}, \frac{25}{49})$ for symmetric case, $Beta(5,2)$ for left skewed case and $Exp(\frac{5}{7})$ for right skewed case. To eliminate the effect of the numerator in the Sharpe ratio, the means of the marginal distributions are considered the same. Table 1 presents the values of the Sharpe ratio. It can be seen that the value of the Sharp ratio decreases with increasing of the dependence of the returns of the portfolio components. At a fixed level of dependence, the Sharp ratio for the case where the marginal distributions are skewed is greater than for the case where the distributions are symmetric.

5 Data Analysis

In this section, we use a real data sets to compare the copula approach and emperical method for calculating the Sharpe ratio. We use three European exchange rates include, GBP/USD, EUR/USD, and CHF/USD and create three portfolios: (1) (GBP/USD, EUR/USD), (2) CHF/USD, EUR/USD and (3) (CHF/USD, GBP/USD). The analyzed period was from the 1st of July of 2010 to the 1st of June 2021 in monthly frequency. Data selected from finance.yahoo.com. To calculate the Sharpe ratio using copula, first a suitable marginal distribution is selected for each return and in the next step, a suitable copula structure will be selected for each pair of portfolio components and the Sharpe ratio is calculated by the formula (2.1). The goodness of fit tests showed that the Logistic distribution is a good fit for GBP/USD and CHF/USD returns and Student-t distribution for EUR/USD returns. The copula goodness-of-fit tests suggest that the Frank copula is the best fit for the returns structure of portfolio 1 and 3 and the Student-t copula is the best fit for the returns structure of portfolio 2. For comparison, the Sharpe ratio of three portfolios was also calculated under the assumption of independence of returns and the usual empirical method. Table 2 shows the result. We can see the overestimation of the Sharpe ratio calculated under the assumption of independence and underestimation with the usual empirical method that is due to the dependence between returns.

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Analysis of Diabetes Using Copula Generalized Additive Regression Model

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Abstract

Copulas are important tools to construct multivariate distributions using their marginals. These functions provide a flexible way to measure the strength of dependency structure among variables. In many cases, the parameters of the marginals and copula are a response related to a set of covariates. Considering the covariates affect in the accuracy of copula parameters estimation. Generalized additive model (GAM) is a flexible models which capture a linear and non-linear relationships between response and explanatory variables. In this paper application of GAM model in copula is investigated. Next, using copula GAM the best model is determined for analyzing diabetic patients in Ilam province.
Keywords: Copulas, Copula Regression, Generalize additive model, Diabetes.

1 Introduction

Copulas are standardized multivariate distributions with uniform margins which represents the dependency structure among variables. Copula was first introduced by [12] and profoundly explained [9]. Copulas have a wide range of applications such as environmental applications ([10]), finance ([2]), economics ([11]), medicine ([8]).

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In many cases, there are situations where the investigated variables are also determined under the influence of covariates. In such situation, estimating the parameters of copula functions without taking into account the effect of these variables causes a waste of information. By considering the marginal distribution and using generalized linear models, the parameters of the marginal distributions can be estimated and based on them, the parameter of the copula function can be estimated ([4]). Regression approach is one of the statistical techniques widely used to study the relationships among variables. This technique has a few assumptions (linearity, normality ...) that should be checked before fitting the model. Copula regression is a general extension of linear regression which the fitting model doesn't require to check the probability distributions. Here, we utilized copula to model the relationship between two correlate responses, which they are related to covariates [5]). In this line, the marginal parameters play appearance in the role of a response variable y related to covariates to be modeled as functions of the explanatory variables. Generalized Additive Model (GAM) is a non-parametric model, and was introduced by [3]. In GAM the smooth function is used to control the linear and nonlinear relationships between response and covariates.

Diabetes is a chronic disease, non-epidemic disease that costs a lot of money in each year. Based on the chronicity and lack of diabetes' definitive remedy, determining the influential factors on diabetes is one important issue to manage of this disease. Because linear function would not assign the effect of predictor on response, GAM is suggested. Two criteria for diagnosing diabetes are Glycosylated Hemoglobin (HbA1c) and Fast Blood Sugar (FBS). Since the analysis of relationship between HbA1c and FBS is one of the important results in diabetes modeling, therefore study of the dependency structure of these factors is demanded. In this paper, using copula additive model, a model is constructed for analyzing diabetes type II. In this way, two factors HbA1c and FBS along with a set of covariates, were selected to analyze the correlation structure of diabetic patients in Ilam province.

This paper is structured as follows: in Chapter 2, GAM model is briefly reviewed. In the Chapter 3 the copula model along with application of GAM in parameters estimation is presented. The copula GAM model is applied to diabetic patients in Ilam province. A Main results of the paper is given in Section 5.

2 Generalize Additive Model

Generalized Additive Model (GAM) is form of the generalized linear model (GLM) model that controls the linear and nonlinear relationships between dependent and independent variables ([3]). In GAM, the smooth function of quantitative variables connected with the link function of dependent variable. The major difference between additive and other models is to use smooth functions in additive model. Regarding the smooth function s(.), the structure of GAM is as $g(\mu_i) = X_i'\beta + \sum_{i=1}^{m} s(x_{ij})$, in which $\mu_i = E(Y_i)$ and X_i' is the i-th predictors. If covariates be qualitative, s(.) be the Identity function and for quantitative covariates are smooth functions. To determine the non-linearity effect of quantitative covariates effective degrees of freedom (edf) can be used. The $edf > 2$ implies high nonlinear significant effect, $1 < edf \leq 2$ shows weak nonlinear and $edf \leq 1$ denotes to linear effect.

2.1 GAM for Location, scale and, shape

Supposed that $f(x;\theta,\phi)$ is belongs to exponential family of distribution with density function

$$f(x;\theta,\phi) = exp(\frac{\theta x - b(\theta)}{a(\phi)} + c(x,\theta))$$

Where a(.), b(.) and c(.) are arbitrary function and and are Location and Scale Parameters, respectively. In Generalized Additive Models for Location, Scale and Shape (GAMLSS) each parameter appearance in the role of a response variable y related to covariates to be modeled as linear/non-linear or smooth functions of the explanatory variables.

$$\begin{cases} \{g_1(\mu) = X_1B_1 + \sum_{i=1}^{k_1} s_{i1}(x_{i1}) \\ g_2(\sigma) = X_2B_2 + \sum_{i=1}^{k_2} s_{i2}(x_{i2}) \\ g_1(\nu) = X_3B_3 + \sum_{i=1}^{k_3} s_{i3}(x_{i3}) \\ g_4(\tau) = X_4B_4 + \sum_{i=1}^{k_4} s_{i4}(x_{i4}). \end{cases} \tag{2.1}$$

Where, $g_i(.)$ is link function and identified based on based on family of marginal distribution.

3 Copula Function

Copulas are multivariate functions that link univariate uniform distribution for construction multivariate distribution. Copulas describe the dependency structure of variables through multivariate distributions. Supposed that H(x,y) be the bivariate distribution with marginals H(x) and G(y), [12] showed there is copula function $C(.)$ such that $H(x,y) = C(F(x), G(y))$. If the marginals be continues, the copula will be unique. Density copula $c(.)$ is related to copula $C(.)$ based on $c(u,v) = \frac{\partial C(u,v)}{\partial u \partial v}$ and using copulas some correlation measures like Spearman and Kendall' τ are obtained $\rho_s = 12\int_0^1\int_0^1[C(u,v) - uv]dudv$ and $\tau_C = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1$, respectively.

Because of a simple mathematical equation and variety of dependence structures based on deference type of copula families, these function became a popular tool for modeling dependencies structure of random variables. Some families of copulas are Clayton, Joe, Gumbel, Ali-Mikhaeel, Frank and Gaussian ([10]).

3.1 Parameter Estimation

Besides of ordinary approaches in estimates the copula parameters like IFM ([9]), in copula Additive Regression Model (GAM) estimation of copula parameter is related to covariates effects. In this way, first, the link function is chooses and using GAM model the parameters are predicted in terms of covariates. Supposed that each marginal has two parameters, the joint cumulative distribution function (cdf) reduced as

$$H(x,y\,|\Theta) = C(F_X(x|\theta_1,\sigma_1), G_Y(y|\theta_2,\sigma_2), \theta) \tag{3.1}$$

where the parameters modeled by 2.1. Using 3.1, he maximum likelihood function obtained by $\ell(\Theta|x,y) = \sum_{i=1}^{n} \log\{c(F_X(x_i|\theta_{1i},\sigma_{1i}), G_Y(y_i|\theta_{2i},\sigma_{2i}),\theta)\}$, Where $\Theta = (\theta_1,\sigma_1,\theta_2,\sigma_2,\theta)$is the vector of marginal and copula parameters.

Because of using non-linear relationships in GAM, to overcome the effect on smooth of nonlinear effect, [7] introduced penalized maximum likelihood function, $\ell_p(\Theta) = \ell(\Theta) - \frac{1}{2}\Theta^T S_\lambda \Theta$, where $S_\lambda = diag(\lambda_{\theta_1} D_{\theta_1}, \lambda_{\theta 2} D_{\theta_2}, \lambda_{\theta_3} D_{\theta_3}, \lambda_{\theta_4} D_{\theta_4})$ and D is a conventional integrated square second derivative spline penalty ([13]).

4 Data Study

This research is a cross-sectional study and performed on diabetes-type II patients referred to the physician's office in Ilam in 2019. One of the diagnostic criteria for diabetes is using of Glycosylated Hemoglobin (Hba1c) and Blood Sugar (FBS). According to the stress and difficulty of each of these two dependent tests, one of the tests can be performed and the amount of the other test can be predicted. Moreover, each of these criteria depends on some covariates which they can be effective in increasing the accuracy of prediction. Here we used copula additive regression to model the structural dependency of HBA1C and FBS where the marginal's parameters are related to covariates via structured additive predictors. The covariates are several risk agents for diabetes-type II and choose based on Can Risk checklist. Considering this checklist the covariates regarded as Gender, Diastolic blood pressure, high blood pressure, daily walking, Age, Systolic blood pressure (Sbp), Mean Corpuscular Volume (MCV), and Body Mass Index (BMI).

The sample size was obtained by equation $n \geq (\frac{2-2\rho^2+\varepsilon}{\varepsilon})(K+1)$ ([1]), in which ρstand for the correlation coefficient among predictors, ε fall within the range of $(0.05\rho^2, 0.2\rho^2)$ and K refers to the number of predictors. Using previous studies $\rho$was considered as 0.77, ε regarded as the middle point of its interval $0.125\rho^2$, and K regarded as 8. By replacing the determined values, the sample size is attained at least 96, where 156 subjects were choosen.

To fit the copula function, first the marginal distribution functions must be identified. Due to the positive support of HbA1c and FBS, Gamble, Gamma, Exponential, Weibull, Logistic and log-Normal distributions were selected. All distributions were fitted to the data where the p-value and Akaike (AIC) corresponding of each distribution summarized in table 1 According to this

Table 1: P-values and AICs of fitting marginal distributions to HbA1c and FBS

	HbA1c		FBS	
Distribution	AIC	P-Value	AIC	P-Value
Gumbel	614.872	0.991	1737.772	0.993
Gamma	552.942	0.757	1696.409	0.470
Exponential	1035.144	0.000	1975.602	0.000
Weibull	579.507	0.468	1695.906	0.308
Logistic	571.035	0.264	1704.341	0.101
Log-Normal	555.213	0.780	1700.553	0.690

table, Gamma distribution with p-value over than 5 % and the minimum values of AIC is relatively better than the other distributions for both marginals. This analysis is conducted through gjrm package in R version 3. 6, where the density of Gamma given by the density of gamma given by $f_Y(y|\mu,\sigma) = \frac{1}{(\sigma^2\mu)^{1/\sigma^2}} \frac{y^{\frac{1}{\sigma^2}-1} e^{-y/\sigma^2\mu}}{\Gamma(1/\sigma^2)}$ with

$E(Y) = \mu$ and $Var(Y) = \sigma^2\mu$. For estimation of the parameters of the marginals GAM model was fitted to the data where the results of fitting on qualitative predictors summarized in table 2. Note that the qualitative predictors enter the model without any changes. According to this

Table 2: Results of fitting GAM for qualitative predictors to HbA1c and FBS

	μ^{HbA1c}		μ^{FBS}		σ^{HbA1c}		σ^{HbA1c}	
	B value	P-value	B-value	P value	B value	P-value	B-value	P value
Constant	9.30	0.000	146.27	0.001	-1.53	0.63	0.99	0.00
Gender	-0.170	0.582	-3.25	0.774	-0.27	0.99	-1.03	0.65
Dbp	-0.170	0.395	1.75	0.724	-	-	-	-
Hbp	0.565	0.037	21.62	0.049	-0.28	0.77	-0.05	0.75
Daily walking	-0.529	0.023	6.72	0.431	0.04	0.99	-0.53	0.73

table Hbp has significant effect on the level of HbA1c and FBS and Daily walking has significant effect on the level of HbA1c .The results of fitting on quantitative predictors summarized in table 3. According to this table, MCV and BMI have significant and linear effect on HbA1c .

Table 3: Results of fitting GAM for quantitative predictors to HbA1c and FBS

	μ^{HbA1c}		μ^{FBS}		σ^{HbA1c}		σ^{FBS}	
	edf	P-value	edf	P value	edf	P-value	edf	P value
s(Age)	0.01	0.582	0.01	0.99	0.01	0.74	0.00	0.94
s(Sbp)	0.01	0.395	0.01	0.99	0.01	0.99	0.00	0.99
s(BMI)	0.565	0.037	0.01	0.50	-	-	-	-
s(MCV)	0.529	0.023	0.01	0.50	-	-	-	-

The positive correlation between HbA1c and FBS (0.561, p-value=0.000) suggests the used copula must cover the upper Frechet-Hoeffding bound which always results maximum positive correlation. So, some copulas are selected which Table 4 presents the AIC for each fitted copula. It can be seen that Frank copula with AIC=2167.83 is the best copula for modeling the structural dependency of diabetes in Ilam province. The following table shows the parameter estimation of this copula function based on covariates. Based on table 5 allvariables have significant effect on copula parameter which Age and Sbp have linear effect ($0 \leq edf < 1$). By using the obtained copula and estimating the marginal distributions, the mean prediction and 95% CI for Hba1c and FBS are shown in the table 6. It can be seen that the high accuracy prediction based on the copula function and the effect of covariates.

Table 4: Results of fitting GAM for qualitative predictors to HbA1c and FBS

	Gaussian	Clayton	Frank	Ali-Mikhaeel	Joe	Placket	Hugaard	Gumbel
AIC	2172.09	2185.28	2167.83	2177.49	2181.75	2167.63	2174.62	2174.62

Table 5: Results of fitting GAM for qualitative and quantitative predictors for Frank parameter

	qualitative			quantitative	
	B value	P-value		edf	P-value
Constant	11.15	0.000	s(Age)	0.52	0.00
Gender	-4.93	0.000	s(Sbp)	0.53	0.00
Dbp	-1.78	0.000			
Hbp	-1.78	0.000			
Daily walking	0.02	0.000			

5 Main results

In this paper Additive regression combined with copula function to prediction the level of Glycosylated Hemoglobin (HbA1c) and Fasting Blood Sugar (FBS). Based on AIC and P-value, among fitted distributions and copulas , Gamma selected as the best distribution for marginals and Frank chosen as a best copula for modeling of diabetic patients in Ilam province. The results of comparing mean prediction of HbA1c and FBS with mean observed showed the copula additive regression could better explained the structural relationship among variables.

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Table 6: mean Observed and prediction of HbA1c and FBS, and corresponding 95% CI based on Frank copula and Gamma marginals

	Mean		95% CI	
	Observed	Predicted	LCI	UCI
HbA1c	8.21	8.19	8.057	8.319
FBS	165.41	166.3	162.891	169.709

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Copula-based reliability analysis of a complex system subject to Wiener degradation process

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Abstract

The complex systems are of great importance in many real situations. Here, a system consisting n elements each having ℓ dependent components is considered, and the reliability of such system is discussed under degradation performance. It is assumed that the degradation of each component follows a Wiener process and the dependence structure within the components is described by a flexible copula-based multivariate model. Also, it is supposed that system has a k-out-of-n structure, and the components of each elements constitute a series system. A simulation study is provided to illustrate how the dependence of components within each element affects system reliability.
Keywords: Copula function, Complex system, Degradation, Wiener process, System reliability.

1 Introduction

One of the most important coherent systems widely discussed in the reliability literature is the k-out-of-n system. This system works if and only if at least k of its elements work. Such systems have

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various applications in engineering such as the multidisplay system in a cockpit and the multipump system in a hydraulic control system. For more details, we refer to [11] and [12].

In many systems, degradation is one of the main causes of system failure. Degradation provides an efficient way for studying some highly reliable systems when observations of failures are rare. In the literature, the degradation of a single component is often modeled using stochastic processes such as the Wiener process. [10, 9], gamma process [13] or Inverse Gaussian (IG) process [4]. Based on a degradation performance, the lifetime of a component is defined as the first passage time at which the degradation measurement reaches or exceeds a predefined threshold. In recent years, utilizing degradation data to predict k-out-of-n system reliability has become more important than ever before. This is because the degradation analysis uses more information than lifetime data analysis, and aims to characterize the underlying failure process. Nezakati and Razmkhah [7] investigated the reliability of a load sharing k-out-of-n degradation system with dependent competing failures. Zhang et al. [14] analysed a degradation-based state reliability modeling for k-out-of-n systems with multiple monitoring positions.

The past related works assumed that the system consists of n individual elements. However, an engineering system may build from elements each containing multiple dependent components. In recent years, the modeling of the dependent components via copula function has received a great deal of attention, mainly due to the flexibility of copula function. Stochastic degradation process models to the multivariate domain using copula function expanded by Fang and Pan [2]. Peng et al. [1] analyzed the copula-based reliability of degrading systems with dependent failures. Recently, Saberzadeh and Razmkhah [8] studied the reliability of some complex system under degradation performance.

In this paper, a degrading complex system consisting of n independent elements each having ℓ dependent components is considered. The components are assumed to constitute a series system, and all elements form a k-out-of-n system. Here, we study the reliability function of such complex system under degradation performance. Toward this end, a Wiener process is considered for the degradation of each component over time. Also, the dependence structure within the components is described by copula function.

The rest of this paper is organized as follows. Section 2 introduces the copula models for multivariate degradation processes. Section 3 describes the complex k-out-of-n:ℓ series systems with dependent degrading components in details. In Section 4, the reliability of the proposed system is derived. Some numerical results are presented in Section 5. In Section 6, some conclusions are stated.

2 Preliminaries

A copula is a function which joins a multivariate distribution function to its one-dimensional marginal distribution functions. Let $\mathbf{X} = (X_1, X_2, ..., X_\ell)^T$ be a ℓ-dimensional random vector with marginal cdfs $F_1(x_1)$, $F_2(x_2)$, ..., $F_p(x_\ell)$ and H be their joint cdf. According to Sklar's theorem [6], there exists a unique copula $C(\cdot)$ such that, for all $x_1, x_2, ..., x_\ell$ in R,

$$H(x_1, x_2, ..., x_\ell) = C(F_1(x_1), F_2(x_2), ..., F_\ell(x_\ell)). \tag{2.1}$$

It states any multivariate distribution can be decomposed into a copula and its marginals. Thus, copula functions offer a much more flexible method to study multivariate distributions. The survival copula, $\bar{C}(\cdot)$, as defined in [5], is given by

$$\bar{C}(u_1, u_2, ..., u_\ell) = 1 + \sum_{k=1}^{\ell} (-1)^k \sum_{1 \le i_1 \le i_2 \le ... \le i_k \le \ell} C_{i_1 i_2 ... i_k}(u_{i_1}, u_{i_2}, ..., u_{i_k}), \tag{2.2}$$

where $C_{i_1 i_2 ... i_k}(u_{i_1}, u_{i_2}, ..., u_{i_k})$ stands for the marginal of $C(\cdot)$ related to a k-combination of $\{1, 2, ..., \ell\}$. In a bivariate case, the survival copula can be expressed as follows

$$\bar{C}(u, v) = 1 - u - v + C(u, v).$$

Thus, the survival function of $\mathbf{X} = (X_1, X_2)^T$ is given by

$$\bar{H}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2) = 1 - F_1(x_1) - F_2(x_2) + H(x_1, x_2).$$

Among all copulas, there is a popular family of copulas called the Archimedean family. This family admits explicit formulas and allows you to model variable dependence through an association parameter. In this paper we consider one of the widely-used copulas in the Archimedean family called Gumbel copula. For ℓ-dimensional case, it is given by

$$C_\lambda(u_1, u_2, ..., u_\ell) = \exp\big\{ - [(-\ln\, u_1)^\lambda + (-\ln\, u_2)^\lambda + ... + (-\ln\, u_\ell)^\lambda]^{1/\lambda} \big\},$$

where $\lambda \in [1, \infty)$ is an association parameter, which is used to measure the dependency between variables.

3 Model description

Consider a system with n elements each containing ℓ components that degrade over time. Suppose that the components of each element are dependent but the elements of system work independently. In such a system, the components are assumed to constitute a series system, and all elements form a k-out-of-n system, that is, the system works if and only if at least k of its elements are functioning. Such a system is called as a complex k-out-of-n:ℓ series system. Assume that the deterioration measures of the hth ($h = 1, 2, ..., \ell$) component of the ith ($i = 1, 2, ..., n$) element can be regarded as a time-dependent stochastic process $X_i^h(t)$ for all $t \ge 0$. We call a component failed (state $= 0$) whenever its degradation reaches or exceeds a threshold level, otherwise it is working (state $= 1$). Suppose the vector $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_\ell)^T$ is the pre-defined thresholds of the components. Therefore, at time t, the hth component of the ith element with degradation $X_i^h(t)$ is in working state, if $X_i^h(t) < \omega_h$, and it is failed if $X_i^h(t) \ge \omega_h$, for $i = 1, 2, ..., n$ and $h = 1, 2, ..., \ell$. Let us denote the state of the ith element at time t by $s_i(t)$. Due to the series structure of components, an element is in working state if all its components are working, i.e.

$$s_i(t) = \begin{cases} 1, & \text{if } X_i^1(t) < \omega_1, X_i^2(t) < \omega_2, ..., X_i^\ell(t) < \omega_\ell, \\ 0, & \text{otherwise.} \end{cases}$$

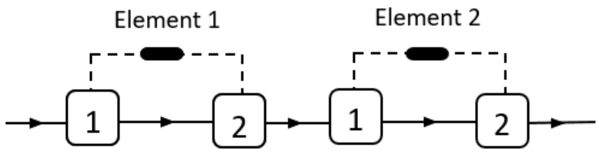

Figure 1: A complex 2-out-of-2:2 series system.

Therefore, the structure function of the complex k-out-of-n:ℓ series system is given by

$$\Phi_t(s_1(t), ..., s_n(t)) = \begin{cases} 1, & \sum_{i=1}^{n} s_i(t) \geq k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Example 1. A complex 2-out-of-2:2 series system can be schematically described in Figure 1. This system has two elements each containing two components. The components are supported by the same source which is seen in this figure as a black item. Therefore, the components of each element are dependent because the working state of components is influenced by the performance of a common source. Each element fails if at least one of its components fail. Furthermore, according to (3.1) the system works when all of its elements work.

From degradation viewpoint, the failure time T of a component is defined as the first time when the degradation process $X(t)$ reaches or exceeds a predetermined threshold ω, i.e.,

$$T = \inf\{t \geq 0;\ X(t) \geq \omega\}.$$

Assume that the degradation of a component follows a Wiener process $\{X(t),\ \ t \geq 0\}$ with $X(t) \sim N\big(\alpha\Lambda(t;\gamma), \beta^2\Lambda(t;\gamma)\big)$, where $\Lambda(t;\gamma)$ is a real-valued function for $t \geq 0$, and γ controls the time scale. Folks and Chhikara [3] proved that the first passage time ($i.e.T$) follows an IG distribution under Wiener process. Thus, the reliability function of a component under Wiener process is obtained as

$$R(t) = P(T > t) = 1 - \Phi\bigg(\frac{\alpha\Lambda(t;\gamma) - \omega}{\beta\sqrt{\Lambda(t;\gamma)}}\bigg) - \exp\Big(\frac{2\alpha\omega}{\beta^2}\Big)\,\Phi\bigg(- \frac{\alpha\Lambda(t;\gamma) + \omega}{\sqrt{\beta\Lambda(t;\gamma)}}\bigg), \tag{3.2}$$

where $\Phi(\cdot)$ is the standard normal cumulaive density function.

4 System reliability

In this section, we investigate the reliability of the complex k-out-of-n:ℓ series systems. It is assumed that the components of each element constitute a series structure. Thus, an element fails when the degradation of just one of its components reaches or exceeds its corresponding threshold level. In fact, for a given threshold vector $\boldsymbol{\omega}$, the lifetime T_i^s of the ith element ($i = 1, 2, ..., n$) is defined as the last instant at which all of the degradation paths has not still reached its threshold level. In other words, we have

$$T_i^s = \sup\{t \geq 0;\ X_i^1(t) < \omega_1, X_i^2(t) < \omega_2, ..., X_i^\ell(t) < \omega_\ell\}.$$

Let $F_{X_i^h(t)}(\cdot, \theta_h)$ shows marginal distribution function of the $X_i^h(t)$ with associated parameter $\theta_h = (\alpha_h, \beta_h, \gamma_h)$, for $h = 1, 2, ..., \ell$ and $i = 1, 2, ..., n$. The reliability of the ith ($i = 1, 2, ..., n$) element may be expressed as

$$\begin{aligned} P(T_i^s > t) &= P(T_i^1 > t, T_i^2 > t, ..., T_i^\ell > t) \\ &= \bar{C}_\lambda\big(F_{T_i^1}(t; \omega_1, \theta_1), F_{T_i^2}(t; \omega_2, \theta_2), ..., F_{T_i^\ell}(t; \omega_\ell, \theta_\ell)\big), \end{aligned} \tag{4.1}$$

where $\bar{C}_\lambda(\cdot)$ is the survival function defined by (2.2) and $F_{T_i^h}(t; \omega_h, \theta_h)$ ($h = 1, 2, \ell$) stands for the the marginal cdf of T_i^h which is derived by (3.2). So, the reliability function of a complex k-out-of-n:ℓ series system is given by

$$\begin{aligned} R^s(t, \boldsymbol{\theta}) &= \sum_{i=k}^{n} \binom{n}{i} \Big(\bar{C}_\lambda\big(F_{T_i^1}(t; \omega_1, \theta_1), F_{T_i^2}(t; \omega_2, \theta_2), ..., F_{T_i^\ell}(t; \omega_\ell, \theta_\ell)\big)\Big)^i \\ &\quad \times \Big(1 - \bar{C}_\lambda\big(F_{T_i^1}(t; \omega_1, \theta_1), F_{T_i^2}(t; \omega_2, \theta_2), ..., F_{T_i^\ell}(t; \omega_\ell, \theta_\ell)\big)\Big)^{n-i}, \end{aligned} \tag{4.2}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_\ell, \lambda)$ is for the vector of model parameters.

5 Sensitive analysis

In this section, we study the sensitivity of the reliability function of a k-out-of-3:ℓ series system with respect to t and λ for $k = 1, 3$ and $\ell = 2, 3$. Toward this end, the power transformation on the time scale $\Lambda(t; \gamma_h) = t^{\gamma_h}$ is considered, and the Gumbel copula with parameter λ is assumed as the dependence structure of the components of each element. The graphs of reliability functions with respect to t are shown in Figure 2 for the parameters of marginal degradation processes $\theta_1 = (0.02, 0.4, 0.5)$, $\theta_2 = (0.02, 2, 1.3)$, $\theta_3 = (0.005, 2, 2)$, the dependence parameter $\lambda = 1.4$, and the degradation thresholds $\boldsymbol{\omega} = (130, 120, 150)$. It is deduced that the reliability function is decreasing in ℓ when other parameters are fixed; moreover, the reliability of a 1-out-of-3:ℓ series system is more than a 3-out-of-3:ℓ series system.

To demonstrate how the dependence parameter effects the reliability function, we provide the plot of the reliability with respect to λ at a fixed point $t = 1000$ in Figure 3. It is observed that the reliability function is increasing in λ.

6 Conclusion

The reliability of a degrading k-out-of-n:ℓ series system was studied under Wiener degradation process. The dependency between components was modelled by Gumbel copula. A sensitivity analysis was done and the behaviour of the reliability function in a special example of the systems was investigated. It was seen that the reliability function increase with respect to dependence parameter, while it is decreasing in both of k and ℓ. The results of this paper may be extended to the following cases:

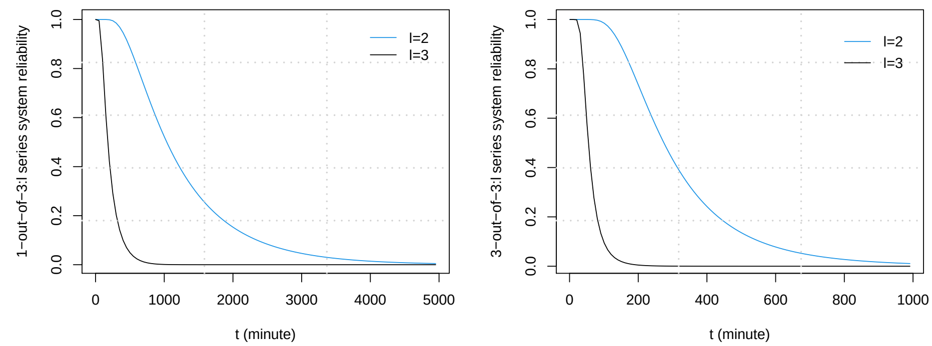

Figure 2: The reliability function of the k-out-of-3:ℓ series system for $k = 1, 3$ and $\ell = 2, 3$. .

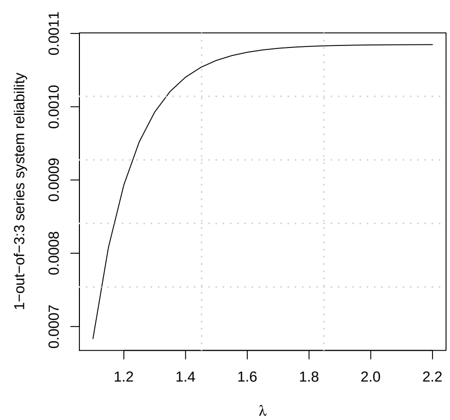

Figure 3: The reliability function of the 1-out-of-3:3 series system at time point $t = 1000$ with respect to λ.

1. Some other copula functions may be used to model the dependency of the components.

2. Other stochastic processes such as gamma or IG processes or even a general path degradation model may be considered instead of Wiener process.

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